REMARKS ON THE COMPLETE SYNCHRONIZATION OF KURAMOTO OSCILLATORS

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Abstract. We present an improved exponential frequency synchronization estimate for globally coupled Kuramoto oscillators. For a sufficiently large coupling, it is numerically observed that Kuramoto oscillators exhibit relaxation toward the phase-locked state, independent of the initial configuration. This phenomenon has never been confirmed analytically in full generality. To date, the analytical treatment of complete frequency synchronization is restricted to initial configurations that are geometrically confined to the half-unit circle. We extend this previous work [7, 8] to a class of initial configurations lying on an arc of length greater than π by exploiting the dynamics of the Kuramoto order parameter in a finite-dimensional setting.

1. Introduction

The purpose of this paper is to continue the work of [7, 8, 16] in seeking a framework for the complete frequency synchronization problem. The synchronization of limit-cycle oscillators has been reported in the literature, from Huygen's pendulum clock to Wiener's α-rhythm theory of brain waves [1, 5, 27, 31, 35]. However, the rigorous model-based analysis of synchronized phenomena began only forty years ago in the pioneering works of Winfree [35] and Kuramoto [21, 22]. In this work, we are only interested in the complete frequency synchronization of Kuramoto oscillators, which denotes the relaxation process of an ensemble of Kuramoto oscillators toward phase-locked states. Kuramoto oscillators can be visualized as point rotors moving on the unit circle $\mathbb{S}^1$. Let $z_i = e^{\sqrt{-1}\theta_i}$ be the position of the i-th rotor on the unit circle. Then, the dynamics of Kuramoto oscillators are governed by the Cauchy problem:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0, \qquad \theta_i(0) = \theta_{i0}, \tag{1.1}$$

where K is the uniform positive coupling strength, and Ω_i represents the intrinsic natural frequency of the i-th oscillator drawn from some distribution function $g = g(\Omega)$. Note that the R.H.S. of system (1.1) is Lipschitz continuous with respect to the state variables and uniformly bounded. Hence, the well-posedness of (1.1) is guaranteed by Cauchy-Lipschitz theory. Note that the R.H.S. of (1.1) is 2π-periodic, so that (1.1) is a dynamical system on $\mathbb{T}^N$. Let $\hat{\Theta} \in \mathbb{R}^N$ be the lift of $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{T}^N$. Then, system (1.1) also induces a dynamical system on the N-dimensional Euclidean space $\mathbb{R}^N$. From now on, we will treat system (1.1) as a system on $\mathbb{R}^N$. For the detailed updated references on the Kuramoto

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model, we refer to review papers [1, 12]

In this paper, we are concerned with the "asymptotic complete frequency synchronization problem" for Kuramoto oscillators. Asymptotic complete frequency synchronization means that all oscillators have the same asymptotic frequency, so that eventually they are all entrained with a fixed shape. Numerical simulations show that, for a sufficiently large coupling, Kuramoto oscillators whose dynamics are governed by (1.1) become entrained, independent of their initial configuration, and the asymptotic phases seem to be a function of system parameters such as Ω_i, K and some averaged information of the initial configuration. Thus, the fine structure of the initial configuration seems to disappear as time goes on. To date, a rigorous justification of complete frequency synchronization is still far from complete. Complete frequency synchronization has been analytically verified for a restricted class of initial configurations under a large coupling strength (see Section 2.2 for details of the state-of-the-art). Therefore, the discrepancy between rigorous results and numerical simulations is rather large, except for identical oscillators. For identical oscillators, complete frequency synchronization has been demonstrated for arbitrary initial configurations [11].

The main results of this paper are twofold. First, we address an exponential complete phase synchronization of identical Kuramoto oscillators for some larger class of initial configurations, including the half circle. Previously, complete phase synchronization (formation of traveling one-point (phase) cluster) has been shown only for initial configurations confined in the half circle in [7]. In contrast, complete frequency synchronization (formation of traveling multi-point cluster with a fixed shape) is verified for any initial configuration based on the gradient flow structure of the Kuramoto model in [11], i.e., as long as the coupling strength is large enough, initial configuration relaxes toward some phase-locked state, and this phase-locked state certainly includes a traveling one-point cluster. However, the gradient flow analysis given in [11, 19] does not yield much information on the relaxation process toward a traveling one-point cluster. Moreover, it also does not characterize admissible initial configurations leading to the complete phase synchronization. Our first result shows that there exists a class of initial configurations that shrink onto the half circle in finite time. This means that the configuration reduces to the existing framework, so we can apply the result in [7] after that time. For this transition estimate, we use two new ideas: the first is an adaptation of the Kuramoto order parameter approach in a finite-dimensional setting. The Kuramoto order parameters consist of the modulus and overall phase of the centroid of the oscillators' positions. These order parameters were used in the synchronization estimate for the mean-field limit. The second idea uses the moving frame centered on the non-constant overall phase. In previous literature [7, 8], the average phase $\theta_c := \dfrac{1}{N}\sum_{i=1}^{N}\theta_i$ was used in the moving frame. Second, we study an exponential complete frequency synchronization of nonidentical oscillators. In this case, unlike that of identical oscillators, the occurrence of complete frequency synchronization independent of the initial configuration has not been analytically verified. To date, the best upper bound for the diameter of admissible initial configurations leading to the complete frequency synchronization is π. Our second result extends this upper bound to $\pi + \varepsilon$, where ε depends on the initial configuration, coupling strength and distribution of natural frequencies (see (4.14)).

Thus, our two results provide new quantitative estimates on the relaxation process of Kuramoto oscillators in a large-coupling regime.

The rest of this paper consists of five sections. In Section 2, we briefly explain the Kuramoto model and derive the dynamics for the order parameters, and review previous literature on the complete frequency synchronization problem. In Section 3, we study the complete frequency synchronization problem for identical oscillators. In Section 4, we extends the complete frequency synchronization problem to nonidentical oscillators. In Section 5, we provide several numerical simulation results, and compare them with the analytical results in Sections 3 and 4. Finally, Section 6 is devoted to a summary of our main results.

2. Preliminaries

In this section, we briefly present a heuristic derivation of the Kuramoto model from the complex Ginzburg–Landau equation, and review previous literature on the complete frequency synchronization problem.

2.1. **From Stuart-Landau oscillators to Kuramoto oscillators.** In this part, we review a heuristic derivation of the Kuramoto model from the dynamics of the linearly coupled Stuart-Landau oscillators in [28] (see also [6, 36] for an alternative derivation from a particle flocking model on the unit sphere $\mathbb{S}^d$ as a special case $d = 1$.

Let $z \in \mathbb{C}$ be the position of the Stuart-Landau oscillator whose dynamics is governed by:

$$\dot{z} = (1 - |z|^2 + \sqrt{-1}\Omega)z, \tag{2.1}$$

where $\Omega \in \mathbb{R}$ is the natural frequency of the Stuart-Landau oscillator. We set $z = re^{\sqrt{-1}\theta}$. Note that (2.1) can be rewritten as follows.

$$\dot{r} = r(1 - r^2), \qquad \dot{\theta} = \Omega. \tag{2.2}$$

Then, it is easy to see that Stuart-Landau oscillator has a stable limit cycle $r = 1$, on which it moves at its natural frequency Ω. We now consider a weakly coupled system of N Stuart-Landau oscillators with a linear all-to-all linear coupling:

$$\frac{dz_j}{dt} = (1 - |z_j|^2 + \sqrt{-1}\Omega_j)z_j + \frac{K}{N}\sum_{i=1}^{N}(z_i - z_j), \tag{2.3}$$

where K is the positive coupling strength. We set z_c to be the centroid of the oscillators:

$$z_c := \frac{1}{N}\sum_{j=1}^{N} z_j.$$

Then, system (2.3) can be rewritten as

$$\frac{dz_j}{dt} = (1 - |z_j|^2 + \sqrt{-1}\Omega_j)z_j + K(z_c - z_j).$$

We now introduce polar forms for z_j and z_c:

$$z_j = r_j e^{\sqrt{-1}\theta_j}, \quad z_c = re^{\sqrt{-1}\phi}.$$

Thus, system (2.3) is equivalent to

$$\begin{aligned}\dot{r}_j &= (1 - r_j^2 - K)r_j + Kr\cos(\phi - \theta_j),\\ \dot{\theta}_j &= \Omega_j + \frac{Kr}{r_j}\sin(\phi - \theta_j).\end{aligned} \tag{2.4}$$

For the limit-cycle oscillators where $r_j = 1$, the second equation of (2.4) becomes the well-known Kuramoto model in terms of r and ϕ:

$$\dot{\theta}_j = \Omega_j + Kr\sin(\phi - \theta_j). \tag{2.5}$$

Note that system (2.5) looks decoupled, but the order parameters r and ϕ are functions of $\theta_j, j = 1, \cdots, N$. Hence, (2.5) corresponds to a rewritten form of the original system (1.1). However, as we can see from (2.5), the effect of neighboring oscillators on the dynamics of a given oscillator is solely through the order parameters r and ϕ. Hence, when r and ϕ are constant, the dynamics of each oscillator is solvable in an exact form (see Appendix of [7]).

System (2.5) has an equilibrium solution if and only if

$$Kr \geq |\Omega_j|, \quad j = 1, \cdots, N,$$

and in this case, we have the stable equilibrium

$$\theta_j = \phi + \sin^{-1}\Big(\frac{\Omega_j}{Kr}\Big) \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

2.2. **Review of the complete frequency synchronization problem.** In this part, we briefly review the state-of-the-art in terms of the complete frequency synchronization problem for the Kuramoto model. For a detailed discussion, we refer readers to survey papers and books [1, 4, 21, 31]. The frequency synchronization problem [1, 9] has been treated using different approaches. Ermentrout [14] found a critical coupling at which all oscillators become phase-locked, independent of their number. The existence of phase locked state and its linear stability have been studied in several papers ([2, 3, 10, 20, 23, 24, 25, 33, 34, 32] using tools such as a Lyapunov functional, spectral graph theory, and control theory. The studies most closely related to this paper are those of Chopra and Spong [8], Choi et al. [7], and Dórfler and Bullo [13]. These papers use the phase-diameter $D(\Theta) := \max_{i,j}|\theta_i - \theta_j|$ as a Lyapunov functional, and study its temporal evolution via Gronwall's inequality. In fact, these papers only deal with initial configurations whose phase-diameter is less than π. To date, π is the best upper bound; if we could extend this upper bound to 2π, it would be possible to rigorously justify the independence of initial configurations observed in numerical simulations. Before we close this section, we recall the most recent result on complete frequency synchronization from [7]. Below, we set

$$D(\Omega) := \max_{1\leq i\leq N}\Omega_i - \min_{1\leq i\leq N}\Omega_i.$$

Theorem 2.1. [7] *Suppose that the coupling strength and initial configuration* Θ_0 *satisfy*

$$0 < D(\Theta_0) =: D_0 < \pi, \qquad K > \frac{D(\Omega)}{\sin D_0}.$$

Then, for any solution $\Theta = (\theta_1, \cdots, \theta_N)$ *to* (1.1) *with initial condition* Θ_0*, there exist positive constants* C_0 *and* Λ *such that*

$$D(\dot{\Theta}(t)) \leq C_0\exp(-\Lambda t), \quad \textit{as } t \to \infty.$$

Remark 2.1. *For identical oscillators $D(\Omega) = 0$, we only need a positive coupling strength $K > 0$. In fact, complete frequency synchronization has been shown in* [11] *for an arbitrary initial configuration with $D(\Theta_0) < 2\pi$. Of course, the synchronization estimate given in* [11] *does not yield the detailed relaxation process toward a phase-locked state.*

3. Basic key estimates

In this section, we study the dynamics of the Kuramoto order parameters and phase-diameter under some a priori assumption. These estimates will be crucial to our complete frequency synchronization estimates in Sections 4 and 5.

3.1. **Dynamics of order parameters.** We introduce Kuramoto order parameters for the finite-dimensional Kuramoto model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i).$$

Recall that, for the phase configuration $\Theta = (\theta_1, \cdots, \theta_N)$, the Kuramoto order parameters r and ϕ are defined by the following relation:

$$re^{\sqrt{-1}\phi} := \frac{1}{N}\sum_{j=1}^{N} e^{\sqrt{-1}\theta_j}. \tag{3.1}$$

Note that r is always bounded, i.e., $0 \le r \le 1$.
It follows from [18] that we have

$$\frac{1}{N}\sum_{j=1}^{N}\cos(\theta_j - \phi) = r, \qquad \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \phi) = 0. \tag{3.2}$$

and the evolutionary system:

$$\begin{aligned} \dot{r} &= -\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \phi)\Big(\Omega_j - Kr\sin(\theta_j - \phi)\Big), \\ \dot{\phi} &= \frac{1}{rN}\sum_{j=1}^{N}\cos(\theta_j - \phi)\Big(\Omega_j - Kr\sin(\theta_j - \phi)\Big). \end{aligned} \tag{3.3}$$

Note that for identical oscillators with $\Omega_j = 0$, it follows from (3.3) that

$$\begin{aligned} \dot{r} &= \frac{Kr}{N}\sum_{j=1}^{N}\sin^2(\theta_j - \phi), \quad t > 0, \\ \dot{\phi} &= -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \phi)\cos(\theta_j - \phi). \end{aligned} \tag{3.4}$$

The monotonicity of r can be easily seen from the first equation of (3.4). Note that the order parameter r is non-decreasing, but may not be strictly increasing; for example, let Θ_0 be the initial configuration such that $m(\neq \frac{N}{2})$ identical oscillators are located at 0 and

$N-m$ are located at π. Then, it is easy to see that this configuration is an equilibrium for (1.1) and

$$z_c = \frac{me^{\sqrt{-1}0} + (N-m)e^{\sqrt{-1}\pi}}{N} = \frac{2m-N}{N} \neq 0, \quad r = \Big|\frac{2m-N}{N}\Big| > 0.$$

Thus, we have

$$r(t) = r(0), \qquad \forall\, t > 0.$$

In the following, we present the dynamics of r for nonidentical oscillators. For positive constants $\alpha, \delta < \frac{1}{2}$, we set r_* and r^* such that

$$\beta_\delta := (1-\delta)\pi, \qquad r_* := \frac{\max_j |\Omega_j|}{\sqrt{\alpha}K \sin\beta_\delta}, \qquad r^* := 1 - \alpha(2 + \sin^2\beta_\delta) > 0. \tag{3.5}$$

For a given configuration $\Theta = (\theta_1, \cdots, \theta_N) \in (\phi - \pi, \phi + \pi]^N$, we set extremal indices M and m:

$$M := \mathrm{argmax}_{1 \le i \le N}(\theta_i - \phi), \qquad m := \mathrm{argmin}_{1 \le i \le N}(\theta_i - \phi).$$

For such M and m, we define the *phase-diameter* $D(\Theta)$ as:

$$D(\Theta) := \theta_M - \theta_m.$$

Below, we use the notation

$$r_0 := r(0). \tag{3.6}$$

Lemma 3.1. *Suppose that the initial configuration and parameters δ, α, and K satisfy*

$$(i)\ \max\{\theta_M(0) - \phi(0), \phi(0) - \theta_m(0)\} \le \beta_\delta, \quad 0 < \delta < \frac{1}{2}.$$
$$(ii)\ 0 < \alpha < \frac{1}{2}, \quad 0 < r_* < r^* := 1 - \alpha(2 + \sin^2\beta_\delta) < 1.$$

Then the following assertions hold.

(1) If $r_* \le r_0 \le r^*$, then the order parameter r is in non-decreasing mode at $t = 0$:

$$\dot r(0) \ge 0.$$

(2) If $r_* < r_0$ and as long as

$$\max_{0 \le s \le t} \max\{\theta_M(s) - \phi(s), \phi(s) - \theta_m(s)\} \le \beta_\delta,$$

we have

$$\min_{0 \le s \le t} r(s) \ge \min\{r_0, r^*\}.$$

Proof. It suffices to show that, as long as $r_* \le r_0 \le r^*$, r is in non-decreasing mode at $t = 0$,

$$\dot r(0) \ge 0.$$

Once we have this, by the exactly same argument, we can show

$$\dot r(s) \ge 0 \qquad 0 \le s \le t, \tag{3.7}$$

if $r_* \le r(s) \le r^*$ is provided. From this fact, the second assertion can be proven as follows :

- Case A $(r_* < r_0 \le r^*)$: Suppose there is a time $s_0 > 0$ such that

$$0 < s_0 \le t, \quad r(s_0) < r_0 = \min\{r_0, r^*\}. \tag{3.8}$$

Define

$$s_1 := \sup_{0\le s\le s_0} \{s : r(s) \ge r_0\}.$$

Since r is continuous function, it is clear that $r(s_1) = r_0$ and $s_1 < s_0$. Now, we choose s_2 satisfying $s_1 < s_2 < s_0$ and

$$r_* \le \min_{s_1\le s\le s_2}\{r(s)\} \le \max_{s_1\le s\le s_2}\{r(s)\} \le r_0 \quad \text{and} \quad r_* < r(s_2) < r_0 = r(s_1).$$

This implies that there should be a time s_3 such that

$$s_1 \le s_3 \le s_2, \quad r_* \le r(s_3) < r_0 \quad \text{and} \quad \dot r(s_3) < 0.$$

However, this is a contradiction to the property (3.7). So there can't be s_0 satisfying (3.8), and hence we have

$$\min_{0\le s\le t} r(s) \ge r_0 = \min\{r_0, r^*\}.$$

- Case B $(r^* < r_0)$: Again, we suppose there is a time s_0 satisfying

$$0 < s_0 \le t, \quad r(s_0) < r^* = \min\{r_0, r^*\}. \tag{3.9}$$

Then, similar to the case A, there should be a time s_1 such that

$$0 < s_1 \le s_0, \quad r_* \le r(s_1) < r^* \quad \text{and} \quad \dot r(s_1) < 0,$$

which is again a contradiction to the fact (3.7). Hence, we have

$$\min_{0\le s\le t} r(s) \ge r^* = \min\{r_0, r^*\}.$$

Now let us prove the first assertion. It follows from (3.3) and the Cauchy–Schwartz inequality that

$$\begin{aligned}
\dot r &= -\frac{1}{N}\sum_{j=1}^N \Omega_j \sin(\theta_j - \phi) + \frac{Kr}{N}\sum_{j=1}^N \sin^2(\theta_j - \phi) \\
&\ge -\frac{1}{N}\Big(\sum_{j=1}^N \Omega_j^2\Big)^{\frac12}\Big(\sum_{j=1}^N \sin^2(\theta_j-\phi)\Big)^{\frac12} + \frac{Kr}{N}\sum_{j=1}^N \sin^2(\theta_j - \phi) \\
&= \frac{K}{\sqrt N}\Big(\sum_{j=1}^N \sin^2(\theta_j-\phi)\Big)^{\frac12}\Big[r\Big(\frac1N\sum_{j=1}^N \sin^2(\theta_j-\phi)\Big)^{\frac12} - \frac{1}{K\sqrt N}\Big(\sum_{j=1}^N \Omega_j^2\Big)^{\frac12}\Big] \\
&\ge \frac{K}{\sqrt N}\Big(\sum_{j=1}^N \sin^2(\theta_j-\phi)\Big)^{\frac12}\Big[r\Big(\frac1N\sum_{j=1}^N \sin^2(\theta_j-\phi)\Big)^{\frac12} - \frac{\max_{1\le j\le N}|\Omega_j|}{K}\Big].
\end{aligned}$$

Here, we used the simple inequality $\Big(\sum_{j=1}^N \Omega_j^2\Big)^{\frac12} \le \sqrt N \max_j |\Omega_j|$.

Suppose that

$$r_* \le r_0 \le r^*. \tag{3.10}$$

We claim:

$$r_0\Big(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_j-\phi)\Big)^{\frac{1}{2}}\Big|_{t=0} - \frac{\max_{1\le j\le N}|\Omega_j|}{K} \ge 0. \tag{3.11}$$

Proof of (3.11): It follows from (3.2) and (3.10) that

$$\begin{aligned}\frac{1}{N}\sum_{j=1}^N \cos(\theta_{j0}-\phi_0)\le r^* &\Longleftrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos(\theta_{j0}-\phi_0)\le Nr^*\\ &\Longleftrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)\le Nr^*-\sum_{j\in I_-}\cos(\theta_{j0}-\phi_0)\\ &\Longrightarrow \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)\le Nr^*-|I_-|\cos\beta_\delta,\end{aligned} \tag{3.12}$$

where $\phi_0=\phi(0)$ and

$$I_+ := \Big\{j : |\theta_{j0}-\phi_0| < \frac{\pi}{2}\Big\}, \qquad I_- := \Big\{j : \frac{\pi}{2}\le |\theta_{j0}-\phi_0|\le \beta_\delta\Big\}.$$

We use (3.12) to obtain

$$\begin{aligned}\sum_{j=1}^N \cos^2(\theta_{j0}-\phi_0) &= \sum_{j\in I_+}\cos^2(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos^2(\theta_{j0}-\phi_0)\\ &\le \sum_{j\in I_+}\cos(\theta_{j0}-\phi_0)+\sum_{j\in I_-}\cos^2(\theta_{j0}-\phi_0)\\ &\le Nr^*-|I_-|\cos\beta_\delta+|I_-|\cos^2\beta_\delta.\end{aligned}$$

On the other hand, we use the inequality $\cos\beta_\delta-\cos^2\beta_\delta\ge 2\cos\beta_\delta$ to derive

$$\begin{aligned}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0) &= N-\sum_{j=1}^N\cos^2(\theta_{j0}-\phi_0)\\ &\ge N-Nr^*+|I_-|(\cos\beta_\delta-\cos^2\beta_\delta)\\ &\ge N-Nr^*+2|I_-|\cos\beta_\delta.\end{aligned}$$

This yields

$$r_0\Big(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0)\Big)^{\frac{1}{2}} \ge r_0\Big(\frac{N-Nr^*+2|I_-|\cos\beta_\delta}{N}\Big)^{\frac{1}{2}}. \tag{3.13}$$

• Case A ($|I_-|>\alpha N$): From our definition of r_*, we have

$$\begin{aligned}r_0\big(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0)\big)^{\frac{1}{2}} &\ge r_*\big(\frac{1}{N}\sum_{j\in I_-}\sin^2(\theta_{j0}-\phi_0)\big)^{\frac{1}{2}}\\ &\ge r_*\sqrt{\alpha}\sin\beta_\delta=\frac{\max_{1\le j\le N}|\Omega_j|}{K}.\end{aligned} \tag{3.14}$$

• Case B ($|I_-| \le \alpha N$): We use definitions of r_*, r^* and (3.13) to obtain

$$
\begin{aligned}
r_0\Big(\frac{1}{N}\sum_{j=1}^N \sin^2(\theta_{j0}-\phi_0)\Big)^{\frac12} &\ge r_*\Big(\frac{N - Nr^* + 2|I_-|\cos\beta_\delta}{N}\Big)^{\frac12} \\
&\ge r_*\big(1 - r^* - 2\alpha\big)^{\frac12} = \frac{\max_{1\le j\le N}|\Omega_j|}{\sqrt{\alpha}K\sin\beta_\delta}\big(\alpha\sin^2\beta_\delta\big)^{\frac12} \\
&= \frac{\max_{1\le j\le N}|\Omega_j|}{K}.
\end{aligned}
\tag{3.15}
$$

Then, it follows from (3.14) and (3.15) that we have the desired estimate (3.11). □

Remark 3.1. *1. Note that, from the proof, if* $r_* < r_0 \le r^*$*, then*

$$
r(t) \ge r_0, \quad t \ge 0.
$$

2. By choosing $\alpha \approx 0$ *and sufficiently large* K*, we can get* $(r_*, r^*) \approx (0,1)$*.*

We next estimate the evolution of the overall phase ϕ in the following lemma.

Lemma 3.2. *Let* ϕ *be the overall phase of the configuration* $\Theta = \Theta(t)$ *whose dynamics is governed by* (1.1)*. Then, we have*

$$
|\dot\phi| \le K(1-r) + \frac{1}{r}\max_{1\le i\le N}|\Omega_i|, \quad t > 0.
$$

Proof. It follows from (3.3) that

$$
\begin{aligned}
\dot\phi &= \frac{1}{rN}\sum_{j=1}^N \cos(\theta_j-\phi)\Big(\Omega_j - Kr\sin(\theta_j-\phi)\Big) \\
&= \frac{1}{rN}\sum_{j=1}^N \Omega_j\cos(\theta_j-\phi) - \frac{K}{N}\sum_{j=1}^N \cos(\theta_j-\phi)\sin(\theta_j-\phi) \\
&=: \mathcal{I}_1 + \mathcal{I}_2.
\end{aligned}
\tag{3.16}
$$

• (Estimate of $\mathcal{I}_1$): We use a rough bound

$$
|\Omega_j\cos(\theta_j-\phi)| \le \max_{1\le j\le N}|\Omega_j|
$$

to obtain

$$
|\mathcal{I}_1| \le \frac{1}{r}\max_{1\le j\le N}|\Omega_j|. \tag{3.17}
$$

• (Estimate of $\mathcal{I}_2$): Below, we provide the upper and lower bounds for $\mathcal{I}_2$.

• Case A (Upper bound): We use (3.2) to obtain

$$
\begin{aligned}
\mathcal{I}_2 &= -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)\cos(\theta_j-\phi)\\
&= -\frac{K}{N}\sum_{j=1}^{N}\Big[(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi)-1)+\cos(\theta_j-\phi)+\sin(\theta_j-\phi)-1\Big]\\
&= -\frac{K}{N}\sum_{j=1}^{N}\underbrace{(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi)-1)}_{\geq 0}-\frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)\\
&\quad -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)+K\\
&\leq -\frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)+K\\
&= K(1-r).
\end{aligned}
\tag{3.18}
$$

• Case B (Lower bound): Similar to Case A, we have

$$
\begin{aligned}
\mathcal{I}_2 &= -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)\cos(\theta_j-\phi)\\
&= -\frac{K}{N}\sum_{j=1}^{N}\Big[(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi)+1)-\cos(\theta_j-\phi)+\sin(\theta_j-\phi)+1\Big]\\
&= -\frac{K}{N}\sum_{j=1}^{N}\underbrace{(\cos(\theta_j-\phi)-1)(\sin(\theta_j-\phi)+1)}_{\leq 0}+\frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)\\
&\quad -\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)-K\\
&\geq \frac{K}{N}\sum_{j=1}^{N}\cos(\theta_j-\phi)-\frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j-\phi)-K\\
&= -K(1-r).
\end{aligned}
\tag{3.19}
$$

Finally, we combine (3.18) and (3.19) to obtain

$$
-K(1-r)\leq \mathcal{I}_2 \leq K(1-r). \tag{3.20}
$$

In (3.16), we combine (3.17) and (3.20) to obtain the desired estimate. □

3.2. **Evolution of phase-diameter.** In this subsection, we provide a decay estimate of the phase-diameter $D(\Theta)$ under a priori condition of fluctuations.

We first remind the Gronwall's inequality whose proof can be found in various places, for example in the Appendix B of [15].

Lemma 3.3 (Gronwall's inequality)**.** *Let $f : [0,T] \mapsto (-\infty,\infty)$ be a nonnegative, differentiable function and let $g, h : [0,T] \to (-\infty,\infty)$ are summable functions.*
(i) If we have the differential inequality

$$f'(t) \le g(t)f(t) + h(t), \quad \forall t \in [0,T],$$

then the following inequality holds

$$f(t) \le f(0)e^{\int_0^t g(s)ds} + \int_0^t h(s)e^{\int_s^t g(\tau)d\tau}ds.$$

(ii) Likewise, if we have

$$f'(t) \ge g(t)f(t) + h(t), \quad \forall t \in [0,T],$$

then the following inequality is true

$$f(t) \ge f(0)e^{\int_0^t g(s)ds} + \int_0^t h(s)e^{\int_s^t g(\tau)d\tau}ds.$$

Lemma 3.4. *For a positive constant $T \in (0,\infty]$, let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to* (1.1) *satisfying the a priori condition:*

$$\beta_T := \max_{0\le\tau\le T} \max\{\theta_M(\tau) - \phi(\tau), \phi(\tau) - \theta_m(\tau)\} < \pi. \tag{3.21}$$

Then, the phase-diameter $D(\Theta)$ satisfies the following lower and upper bounds: For any $0 < t < T$,

$$(i)\ D(\Theta_0)e^{-K\int_0^t r(s)ds} - D(\Omega)\int_0^t e^{-K\int_s^t r(\tau)d\tau}ds \le D(\Theta(t)).$$
$$(ii)\ D(\Theta(t)) \le D(\Theta_0)e^{-K\frac{\sin\beta_T}{\beta_T}\int_0^t r(s)ds} + D(\Omega)\int_0^t e^{-K\frac{\sin\beta_T}{\beta_T}\int_s^t r(\tau)d\tau}ds.$$

Proof. (i) (Lower bound estimate): We use equation (2.5) to derive

$$\begin{aligned} \dot{D}(\Theta) &= \dot{\theta}_M - \dot{\theta}_m \\ &= \Omega_M - \Omega_m - Kr\left(\sin(\theta_M - \phi) - \sin(\theta_m - \phi)\right) \\ &\ge \Omega_M - \Omega_m - Kr(\theta_M - \theta_m) \\ &\ge -D(\Omega) - KrD(\Theta), \end{aligned} \tag{3.22}$$

where the first inequality comes from the fact that

$$\sin x \begin{cases} \le x, & if \quad x \ge 0, \\ \ge x, & if \quad x \le 0. \end{cases}$$

Then, Gronwall's lemma for (3.22) yields the desired lower bound estimate for $D(\theta)$.

(ii) (Upper bound estimate): We first note that, under the a priori condition (3.21), i.e.,

$$-\beta_T \le \theta_m - \phi \le 0 \le \theta_M - \phi \le \beta_T, \quad \text{for some } 0 < \beta_T < \pi,$$

we have

$$\begin{aligned} \sin(\theta_M - \phi) - \sin(\theta_m - \phi) &\ge \frac{\sin\beta_T}{\beta_T}(\theta_M - \phi) - \frac{\sin\beta_T}{\beta_T}(\theta_m - \phi) \\ &= \frac{\sin\beta_T}{\beta_T}(\theta_M - \theta_m). \end{aligned} \tag{3.23}$$

Then, we use (3.23) to obtain

$$
\begin{aligned}
\dot{D}(\Theta) &= \Omega_M - \Omega_m - Kr\left(\sin(\theta_M - \phi) - \sin(\theta_m - \phi)\right) \\
&\le \Omega_M - \Omega_m - Kr\frac{\sin\beta_T}{\beta_T}(\theta_M - \theta_m) \\
&= \Omega_M - \Omega_m - Kr\frac{\sin\beta_T}{\beta_T}D(\Theta) \\
&\le D(\Omega) - Kr\frac{\sin\beta_T}{\beta_T}D(\Theta).
\end{aligned} \tag{3.24}
$$

Now, (3.24) and Gronwall's lemma imply the desired estimate. □

4. Complete frequency synchronization

In this section, we extend the frequency synchronization estimate to initial configurations whose diameter is larger than π.

4.1. **Identical oscillators.** Consider Kuramoto oscillators with

$$\Omega_i = 0, \quad 1 \le i \le N.$$

Although the results in [11, 20, 26, 29, 30] establish complete frequency synchronization and complete phase synchronization for an arbitrary initial configuration and almost all initial configuration, respectively, we do not have detailed information about the relaxation process and structure of phase-locked states. Of course, it is known that the only stable phase-locked state corresponds to complete phase synchronization consisting of a single phase. In the following, we study the detailed relaxation process by investigating the dynamics of the order parameters instead of the phase-diameter.

Lemma 4.1. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to* (1.1) *with initial data Θ_0 satisfying $r_0 > 0$. Then, we have*

$$\lim_{t\to\infty} \sin(\theta_i(t) - \phi(t)) = 0, \quad i = 1, \cdots, N.$$

Proof. It follows from $(3.4)_1$ that we have

$$r(t) = r_0 \exp\Big(\int_0^t g(\tau)d\tau\Big), \qquad g(\tau) := \frac{K}{N}\sum_{j=1}^N \sin^2(\phi(\tau) - \theta_j(\tau)).$$

Because r is bounded from above by 1, the nonnegative function g belongs to $L^1(0,\infty)$, i.e.,

$$\int_0^\infty g(t)dt = \frac{K}{N}\sum_{j=1}^N \int_0^\infty \sin^2(\phi(t) - \theta_j(t))dt < \infty. \tag{4.1}$$

We claim:

$$\lim_{t\to\infty} \sin^2(\theta_j(t) - \phi(t)) = 0, \qquad j = 1, \cdots, N.$$

Proof of claim: Suppose not, i.e., there exist $j, \nu_0 > 0$ and an increasing sequence of times $\{t_n\}_{n=1}^\infty$ such that

$$\sin^2(\phi(t_n) - \theta_j(t_n)) > \nu_0 \qquad \text{and} \qquad \lim_{n\to\infty} t_n = \infty.$$

On the other hand, note that

$$\begin{aligned}\frac{d}{dt}\sin^2(\phi(t)-\theta_j(t)) &= 2\sin(\phi(t)-\theta_j(t))\cos(\phi(t)-\theta_j(t))(\dot\phi(t)-\dot\theta_j(t))\\ &= (\dot\phi(t)-\dot\theta_j(t))\sin(2(\phi(t)-\theta_j(t))).\end{aligned}$$

We use the above relation and Lemma 3.2 to obtain

$$\left|\frac{d}{dt}\sin^2(\phi(t)-\theta_j(t))\right| \le (|\dot\phi(t)|+|\dot\theta_j|) \le K-Kr+Kr=K. \tag{4.2}$$

We use the bound on the derivative in (4.2) to derive

$$\sin^2(\phi(t_n)-\theta_j(t_n))>\nu_0 \quad\Longrightarrow\quad \sin^2(\phi(t)-\theta_j(t))\ge\frac{\nu_0}{2} \qquad t\in\left[t_n-\frac{\nu_0}{2K},t_n+\frac{\nu_0}{2K}\right]. \tag{4.3}$$

Possibly extracting a subsequence(we abuse the notation), we may assume

$$t_{n+1}-t_n>\frac{\nu_0}{K} \qquad \forall n=1,2\dots. \tag{4.4}$$

Combining (4.1), (4.3) and (4.4), we derive a contradiction:

$$\int_0^\infty \sin^2(\phi(t)-\theta_j(t))dt \ge \sum_{n=1}^\infty\int_{t_n-\nu_0/2K}^{t_n+\nu_0/2K}\sin^2(\phi(t)-\theta_j(t))dt \ge \sum_{n=1}^\infty\frac{\nu_0^2}{2K}=\infty.$$

□

In the following corollary, for any initial configuration, we show the complete frequency synchronization, and characterize the phase-locked states for identical Kuramoto oscillators.

Corollary 4.1. *Let $\Theta=(\theta_1,\cdots,\theta_N)$ be a solution to* (1.1) *with initial data Θ_0. Then, we have the following assertions:*

(1) If $r_0>0$, then we have a dichotomy:

$$\lim_{t\to\infty}|\theta_j-\phi|=0 \quad or \quad \pi, \quad for\ all\ j=1,\cdots,N.$$

(2) If $r_0=0$, then the initial configuration Θ_0 is the equilibrium solution to (1.1) *with $\Omega_i=0,\quad 1\le i\le N$.*

Proof. The first assertion follows from Lemma 4.1. Now we focus on the second assertion. Suppose that initial data Θ_0 and natural frequencies Ω_j satisfy

$$r_0=0,\quad \Omega_j=0,\quad 1\le j\le N.$$

We claim:

$$\Theta(t):=\Theta_0, \qquad \forall t>0,$$

is the equilibrium solution of (1.1). First of all, it is clear that

$$\dot\theta_j(t)=0, \qquad 1\le j\le N,\quad \forall t>0. \tag{4.5}$$

On the other hand, for all $t>0$, we have

$$\begin{aligned}\Omega_i+\frac{K}{N}\sum_{j=1}^N\sin(\theta_j(t)-\theta_i(t)) &= \frac{K}{N}\sum_{j=1}^N\sin(\theta_j(0)-\theta_i(0))\\ &= Kr_0\sin(\phi(0)-\theta_j(0)),\end{aligned} \tag{4.6}$$

where the second equality follows from (2.5). Since $r_0 = 0$, Equation (4.6) implies

$$\Omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j(t) - \theta_i(t)) = 0, \quad \forall t > 0. \tag{4.7}$$

Combining (4.5) and (4.7), we verified $\Theta(t) := \Theta_0$ is the solution to (1.1) with Ω_i, $1 \le i \le N$. □

We next present admissible initial configurations that relax to the complete phase configuration exponentially fast. For the relaxation estimate, we split our analysis into two steps. First, we show that, as long as the extremal fluctuations $\theta_M - \phi$ and $\phi - \theta_m$ are less than π, the phase-diameter decays exponentially fast to zero. Second, we identify a class of initial phases whose evolution guarantees the a priori condition.

We are now ready to provide an exponential frequency synchronization for some initial configurations whose diameters are larger than π. Our strategy is as follows. We first show that the initial configuration evolves to a configuration whose diameter is less than π in finite time (entrance time), and then, we apply Theorem 2.1 with this intermediate configuration as new initial data at this entrance time. Below, we set

$$\varepsilon_1(r_0, \delta) := \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{2(1-r_0)}{\pi r_0}\cdot\frac{\beta_\delta}{\sin\beta_\delta}}.$$

Lemma 4.2. *Suppose that, for $0 < \delta < \frac{1}{2}$, the initial configuration satisfies*

$$r_0 > 0, \qquad \max_{1\le i\le N}|\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_1(r_0, \delta). \tag{4.8}$$

Then, for any solution $\Theta = (\theta_1, \cdots, \theta_N)$ of (1.1), there exists $t_e > 0$ such that

$$(\theta_M - \theta_m)(t_e) < \pi.$$

More precisely, we can choose t_e as follows:

$$t_e = \Big(\frac{2\beta_\delta}{\pi K r_0 \sin\beta_\delta}\Big)\cdot\Big(\frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{2(1-r_0)\beta_\delta}{\pi r_0\sin\beta_\delta}}\Big).$$

Proof. We use a bootstrapping argument as follows. We set

$$\mathcal{T} := \{t \in [0,\infty) \ : \ \max_{0\le\tau\le t}\max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \pi\}, \quad \mathcal{T}^* := \sup\mathcal{T}.$$

Then, it follows from (4.8) that:

$$\max\{(\theta_M - \phi)(0), (\phi - \theta_m)(0)\} < \frac{\pi}{2} + \varepsilon_1,$$

and from the continuities of $\theta_M - \phi$ and $\phi - \theta_m$ that there exists $t' > 0$ such that

$$\max_{0\le\tau\le t'}\max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \pi, \quad t' \in \mathcal{T}.$$

- (Rough estimate for $D(\Theta)$): We use the lower bound of $r(t) \ge r_0,\ t \le t'$, Lemma 3.2, and

$$\dot\theta_M = -Kr\sin(\theta_M - \phi) \quad \text{and} \quad \dot\theta_m = Kr\sin(\phi - \theta_m)$$

to derive

$$
\begin{aligned}
&\frac{d}{dt}(\theta_M-\phi)\le -Kr\sin(\theta_M-\phi)+K(1-r)\le K(1-r)\le K(1-r_0),\\
&\frac{d}{dt}(\phi-\theta_m)\le -Kr\sin(\phi-\theta_m)+K(1-r)\le K(1-r)\le K(1-r_0).
\end{aligned}
\tag{4.9}
$$

Then, thanks to (4.9), we will choose $t_e \in \mathcal{T}$ and ε_1 such that, for $t\in(0,t_e)$, we have

$$
\begin{aligned}
\max_{0\le t\le t_e}\max\{\theta_M-\phi,\phi-\theta_m\} &\le \max\{(\theta_M-\phi)(0),(\phi-\theta_m)(0)\}+K(1-r_0)t_e\\
&\le \frac{\pi}{2}+\varepsilon_1+K(1-r_0)t_e\le\beta_\delta<\pi.
\end{aligned}
\tag{4.10}
$$

• (Refined estimate for $D(\Theta)$): Under the rough estimate (4.10), which satisfies the a priori assumption (3.21), we can apply Lemma 3.3 and $r\ge r_0$ to obtain

$$
\begin{aligned}
(\theta_M-\theta_m)(t_e) &\le (\theta_M-\theta_m)(0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}\int_0^{t_e} r(\tau)d\tau}\\
&\le (\theta_M-\theta_m)(0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0t_e}\\
&\le (\pi+2\varepsilon_1)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0t_e}.
\end{aligned}
\tag{4.11}
$$

In (4.10) and (4.11), we need to choose ε_1 and t_e to satisfy

$$
\frac{\pi}{2}+\varepsilon_1+K(1-r_0)t_e\le\beta_\delta \quad\text{and}\quad (\pi+2\varepsilon_1)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0t_e}<\pi.
\tag{4.12}
$$

We next determine explicit functional forms for t_e and ε_1 satisfying relations (4.12). For definiteness, we will look for t_e and ε_1 satisfying the coupled relations:

$$
\begin{aligned}
&\frac{\pi}{2}+\varepsilon_1+K(1-r_0)t_e=(1-\delta)\pi=\beta_\delta \quad\text{and}\\
&\pi+2\varepsilon_1=\pi\Big(1+K\frac{\sin\beta_\delta}{\beta_\delta}r_0t_e\Big).
\end{aligned}
\tag{4.13}
$$

Because $1+x<e^x,\quad x>0$, once we find t_e and ε_1 satisfying (4.13), the pair (t_e,ε_1) also satisfies relation (4.12). From the second equation of (4.13), we have

$$
t_e=\frac{2\beta_\delta}{\pi K r_0\sin\beta_\delta}\varepsilon_1,
$$

and we substitute this relation in the first equation of (4.13) to find

$$
\varepsilon_1=\frac{\beta_\delta-\frac{\pi}{2}}{1+\frac{2(1-r_0)\beta_\delta}{\pi r_0\sin\beta_\delta}} \quad\text{and}\quad t_e=\Big(\frac{2\beta_\delta}{\pi K r_0\sin\beta_\delta}\Big)\cdot\Big(\frac{\beta_\delta-\frac{\pi}{2}}{1+\frac{2(1-r_0)\beta_\delta}{\pi r_0\sin\beta_\delta}}\Big).
$$

□

Theorem 4.1. *Suppose that the initial configuration and* $0<\delta<\frac{1}{2}$ *satisfy*

$$
r_0>0,\qquad \max_{1\le i\le N}|\theta_{i0}-\phi_0|<\frac{\pi}{2}+\frac{\beta_\delta-\frac{\pi}{2}}{1+\frac{2(1-r_0)}{\pi r_0}\cdot\frac{\beta_\delta}{\sin\beta_\delta}},
$$

and let $\Theta=(\theta_1,\cdots,\theta_N)$ *be a solution to (1.1) with initial data* Θ_0*. Then, there exists positive constants* C *and* Λ *such that*

$$
D(\Theta(t))\le Ce^{-\Lambda t},\quad \textit{as } t\to\infty.
$$

Proof. Let Θ be the solution to (1.1) with initial data satisfying conditions (4.8). Then, it follows from Lemma 4.2 that there exists a finite time $t_e > 0$ such that

$$D(\Theta(t_e)) < \pi.$$

Thus, we can apply Theorem 2.1 with initial data $\Theta(t_e)$ after $t > t_e$ to derive the desired exponential frequency synchronization. □

Remark 4.1. *The complete phase synchronization estimates have been extensively studied in literature* [20, 26, 29, 30] *of control theory based on the gradient flow structure of the Kuramoto model and LaSalle's invariance principle, which establishes the complete phase synchronization for almost all initial configuration. However, this analysis does not yield the information on the basin of phase synchronization (see Theorem 5.1 in* [12]*). In contrast, the result in Theorem 4.1 describes the proper subset of basin of phase synchronization.*

4.2. **Nonidentical oscillators.** In this part, we study complete frequency synchronization for nonidentical oscillators by analyzing the dynamics of the order parameters r and ϕ introduced in Section 3. We set

$$\varepsilon_2(r_0, \delta, K, \{\Omega_i\}) := \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{K(1-r_0)+\left(1+\frac{1}{r_0}\right)\max_j|\Omega_j|}{\frac{\pi K r_0 \sin\beta_\delta}{2\beta_\delta} - \frac{D(\Omega)}{2}}}. \tag{4.14}$$

Lemma 4.3. *Suppose that the initial configuration Θ_0 and coupling strength K satisfy*

$$(i)\ r_* < r_0 \le r^*, \qquad \max_{1\le i\le N} |\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_2,$$
$$(ii)\ K > \max\left\{ \frac{\max_j |\Omega_j|}{[1-\alpha(2+\sin^2\beta_\delta)][\sqrt{\alpha}\sin\beta_\delta]}, \frac{\beta_\delta D(\Omega)}{\pi r_0 \sin\beta_\delta} \right\}.$$

Then, there exists a finite time $t_e \in (0,\infty)$ such that

$$(\theta_M - \theta_m)(t_e) < \pi.$$

Proof. We set

$$\mathcal{T} := \{t \in [0,\infty) \ :\ \max_{0\le\tau\le t} \max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \pi\}, \quad T^* := \sup \mathcal{T}.$$

Then, it follows from the initial condition:

$$\max\{(\theta_M - \phi)(0), (\phi - \theta_m)(0)\} < \frac{\pi}{2} + \varepsilon_2$$

and the continuities of $\theta_M - \phi$ and $\phi - \theta_m$ that there exists $t' > 0$ such that

$$\max_{0\le\tau\le t'} \max\{(\theta_M - \phi)(\tau), (\phi - \theta_m)(\tau)\} < \beta_\delta, \quad t' \in \mathcal{T}.$$

• Step A: By the assumption on the initial configuration Θ_0, we have

$$\max\{(\theta_M - \phi)(0), (\phi - \theta_m)(0)\} \le \frac{\pi}{2} + \varepsilon_2.$$

On the other hand, we use

$$\frac{d\theta_M}{dt} = \Omega_M - Kr\sin(\theta_M - \phi), \qquad \frac{d\theta_m}{dt} = \Omega_m - Kr\sin(\theta_m - \phi)$$

and Lemma 3.2 to obtain

$$
\begin{aligned}
\frac{d}{dt}(\theta_M - \phi) &\le \Omega_M - Kr\sin(\theta_M - \phi) + K(1-r) + \frac{\max_j |\Omega_j|}{r} \\
&\le K(1-r) + \left(1 + \frac{1}{r}\right)\max_j |\Omega_j|, \\
\frac{d}{dt}(\phi - \theta_m) &\le K(1-r) + \left(1 + \frac{1}{r}\right)\max_j |\Omega_j|.
\end{aligned} \tag{4.15}
$$

Because $r_* < r_0 \le r^*$, it follows from Lemma 3.1 that

$$
r(t) \ge r_0, \quad 0 \le t \le t'. \tag{4.16}
$$

Then, (4.15) and (4.16) imply that, for $0 < h < t'$,

$$
\max_{0\le t\le h}\{\theta_M - \phi, \phi - \theta_m\} \le \frac{\pi}{2} + \varepsilon_2 + \left[K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|\right] h. \tag{4.17}
$$

As long as

$$
\frac{\pi}{2} + \varepsilon_2 + \left(K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|\right) h \le \beta_\delta < \pi, \tag{4.18}
$$

we can also use (3.24) to obtain

$$
\begin{aligned}
\frac{dD(\Theta)}{dt} &= \dot\theta_M - \dot\theta_m \\
&= \Omega_M - \Omega_m - Kr\left(\sin(\theta_M - \phi) - \sin(\theta_m - \phi)\right) \\
&\le D(\Omega) - Kr\frac{\sin\beta_\delta}{\beta_\delta}D(\Theta).
\end{aligned}
$$

This yields

$$
\begin{aligned}
D(\Theta(h)) &\le D(\Theta_0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}\int_0^h r(s)ds} + D(\Omega)\int_0^h e^{-K\frac{\sin\beta_\delta}{\beta_\delta}\int_s^h r(\tau)d\tau} ds \\
&\le D(\Theta_0)e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 h} + D(\Omega)\int_0^h e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0(h-s)} ds \\
&\le \left[D(\Theta_0) + \frac{D(\Omega)\beta_\delta}{Kr_0\sin\beta_\delta}\left(e^{K\frac{\sin\beta_\delta}{\beta_\delta}r_0 h} - 1\right)\right] e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 h}.
\end{aligned} \tag{4.19}
$$

• Step B (Determination of t_e and ε_2): It follows from (4.17), (4.18), and (4.19) that, if we can determine t_e and ε_2 to satisfy

$$
\begin{aligned}
&\frac{\pi}{2} + \varepsilon_2 + \left(K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|\right) t_e \le \beta_\delta, \\
&D(\Theta(t_e)) \le \left[D(\Theta_0) + \frac{D(\Omega)\beta_\delta}{Kr_0\sin\beta_\delta}\left(e^{K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e} - 1\right)\right] e^{-K\frac{\sin\beta_\delta}{\beta_\delta}r_0 t_e} < \pi,
\end{aligned} \tag{4.20}
$$

then we are done. For this, we claim that the solution to the system:

$$
\begin{aligned}
&\frac{\pi}{2} + \varepsilon_2 + \left(K(1-r_0) + \left(1 + \frac{1}{r_0}\right)\max_j |\Omega_j|\right) t_e = \beta_\delta, \\
&2\varepsilon_2 = \left(\pi - \frac{\beta_\delta D(\Omega)}{Kr_0\sin\beta_\delta}\right)\left(K\frac{\sin\beta_\delta}{\beta_\delta}r_0\right) t_e
\end{aligned} \tag{4.21}
$$

satisfies (4.20). Below, we will prove our claim.

Suppose that (t_e, ε_2) is a solution of system (4.21). Then, we use $D(\Theta_0) < \pi + 2\varepsilon_2$ to see that

$$
\begin{aligned}
2\varepsilon_2 = &\left(\pi - \frac{\beta_\delta D(\Omega)}{Kr_0 \sin\beta_\delta}\right)\left(K\frac{\sin\beta_\delta}{\beta_\delta} r_0\right) t_e \\
&\Longrightarrow 2\varepsilon_2 < \left(\pi - \frac{\beta_\delta D(\Omega)}{Kr_0 \sin\beta_\delta}\right)\left(e^{K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1\right) \\
&\Longrightarrow \pi + 2\varepsilon_2 + D(\Omega)\frac{\beta_\delta}{Kr_0\sin\beta_\delta}(e^{K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1) < \pi e^{K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} \\
&\Longrightarrow \left(\pi + 2\varepsilon_2 + D(\Omega)\frac{\beta_\delta}{K\sin\beta_\delta r_0}(e^{K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1)\right) e^{-K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} < \pi.
\end{aligned} \tag{4.22}
$$

On the other hand, it follows from (4.19) with $h = t_e$, (4.22) and $D(\Theta_0) < \pi + 2\varepsilon_2$ that we have the second inequality of (4.20):

$$
\begin{aligned}
D(\Theta(t_e)) &\leq \left(D(\Theta_0) + D(\Omega)\frac{\beta_\delta}{Kr_0\sin\beta_\delta}(e^{K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1)\right) e^{-K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} \\
&< \left(\pi + 2\epsilon + D(\Omega)\frac{\beta_\delta}{Kr_0\sin\beta_\delta}(e^{K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} - 1)\right) e^{-K\frac{\sin\beta_\delta}{\beta_\delta} r_0 t_e} \\
&< \pi.
\end{aligned}
$$

Note that system (4.21) admits the following solutions:

$$
\begin{aligned}
&t_e = \frac{2\varepsilon_2}{\left(\pi - \frac{\beta_\delta D(\Omega)}{Kr_0\sin\beta_\delta}\right)\left(K\frac{\sin\beta_\delta}{\beta_\delta} r_0\right)} \quad \text{and} \\
&\varepsilon_2\left(1 + \frac{\left(K(1-r_0) + \left(1+\frac{1}{r_0}\right)|\Omega|\right)}{\left(\pi - \frac{\beta_\delta D(\Omega)}{Kr_0\sin\beta_\delta}\right)\left(K\frac{\sin\beta_\delta}{2\beta_\delta} r_0\right)}\right) = \beta_\delta - \frac{\pi}{2}.
\end{aligned} \tag{4.23}
$$

□

Before we state our second main result of this paper, we recall several parameters to be used in the statement of main result. Let α and δ be positive constants in $(0, \frac{1}{2})$ satisfying the relation:

$$1 - \alpha(2 + \sin^2(1-\delta)\pi) > 0.$$

For notational simplicity, we set

$$\beta_\delta := (1-\delta)\pi, \qquad r_* := \frac{\max_j |\Omega_j|}{\sqrt{\alpha} K \sin\beta_\delta}, \qquad r^* := 1 - \alpha(2 + \sin^2\beta_\delta).$$

$$r_0 := r(0), \qquad \varepsilon_2(r_0, \delta, K, \{\Omega_i\}) := \frac{\beta_\delta - \frac{\pi}{2}}{1 + \frac{K(1-r_0) + \left(1+\frac{1}{r_0}\right)\max_j|\Omega_j|}{\frac{\pi K r_0 \sin\beta_\delta}{2\beta_\delta} - \frac{D(\Omega)}{2}}}.$$

We now state our second main result on the complete frequency synchronization of non-identical oscillators.

Theorem 4.2. *Suppose that the initial configuration Θ_0 and coupling strength K satisfy*

$$(i)\ r_* < r_0 \leq r^*, \qquad \max_{1\leq i\leq N} |\theta_{i0} - \phi_0| < \frac{\pi}{2} + \varepsilon_2.$$

$$(ii)\ K > \max\left\{\frac{\max_j|\Omega_j|}{[1-\alpha(2+\sin^2\beta_\delta)][\sqrt{\alpha}\sin\beta_\delta]}, \frac{\beta_\delta D(\Omega)}{\pi r_0 \sin\beta_\delta}\right\}.$$

Then, the exponential frequency synchronization holds.

Proof. It follows from Lemma 4.3 that there exists $t_e \in (0, \infty)$ such that

$$D(\Theta(t_e)) < \pi.$$

We can apply Theorem 2.1 for the configuration at $t = t_e$ as a new initial configuration to derive the desired exponential frequency synchronization. □

Remark 4.2. *The conditions on initial configurations and coupling strength in Theorem 4.2 are not necessary conditions as can be seen in* [17] *where the relaxation toward the phase-locked state can be algebraic depending on the relation between the coupling strength and natural frequency diameter. Thus, our conditions on initial configurations and coupling strength are sufficient conditions to get the fast (exponential) frequency synchronization.*

5. Numerical simulations

In this section, we provide several numerical examples of the results of Section 4. For the numerical simulations, we employed the fourth-order Runge–Kutta method with a time step of $h = 0.01$.

5.1. **Identical oscillators.** For the numerical simulation illustrated in Figure 1, we set

$$K = 1, \quad N = 200, \quad \Omega_i = 0, \quad \delta = 0.2, \quad \beta_\delta = 0.8\pi \approx 2.5133,$$

and chose initial phases $\theta_{i0},\ i = 1, \cdots, 200$ at random from the symmetric interval $[-0.58\pi, 0.58\pi]$ around 0 so that

$$r_0 \approx 0.5632, \qquad \varepsilon_1 \approx 0.3029.$$

Note that, as $1.8221 \approx 0.58\pi < \frac{\pi}{2} + \varepsilon_1 \approx 1.8737$, our initial configuration satisfies the condition in Lemma 4.2 and Theorem 4.1:

$$\theta_{i0} \in [-\frac{\pi}{2} - \varepsilon_1, \frac{\pi}{2} + \varepsilon_1] \quad \text{for all } i = 1, \cdots, 200.$$

Because the initial phase-diameter $D(\Theta_0) \approx 3.6397$ is strictly larger than π, the previous result in Theorem 2.1 cannot be applied. Instead, Theorem 4.1 can be applied to obtain the complete frequency synchronization estimate in Figure 1.

By Lemma 4.2, we have

$$t_e \approx 1.4641.$$

In Figures 1(b) and 1(d), we can see that the diameter of the oscillators at t_e is less than π:

$$D(\Theta(t_e)) \approx 1.6082 < \pi.$$

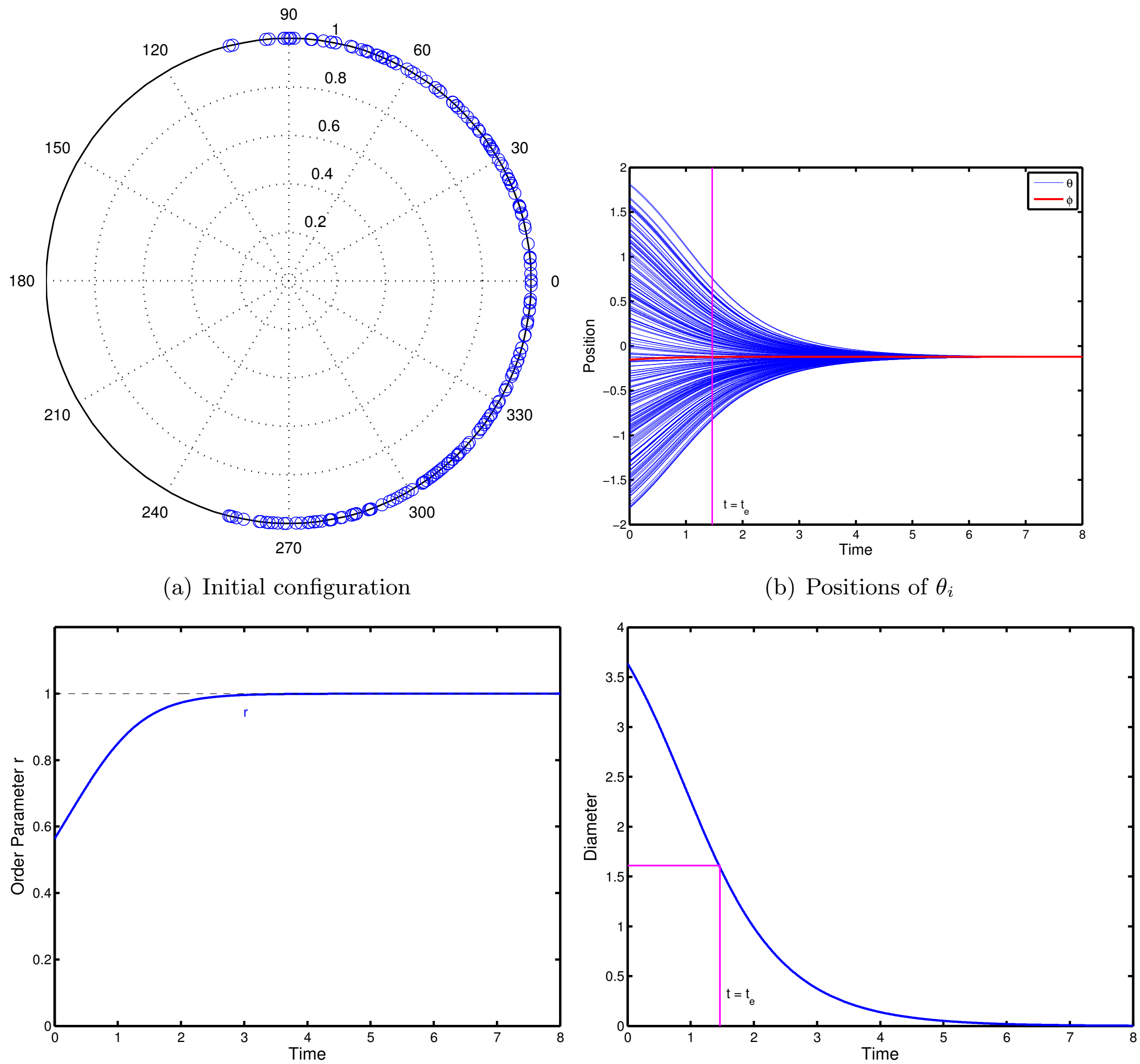
This fact is substantiated by Lemma 4.2. After $t = t_e$, the dynamics follows the previous result in [7], which is vindicated by Theorem 4.1.

5.2. **Nonidentical oscillators.** In this part, we establish a numerical example for the nonidentical oscillator model. For the simulation, we set

$$K = 1.5, \quad N = 200, \quad \alpha = 0.1, \quad \delta = 0.25 \quad \text{and} \quad \beta_\delta = 0.75\pi.$$

Initial phases and natural frequencies were chosen at random from the symmetric intervals $[-0.55\pi, 0.55\pi]$ and $[-0.1, 0.1]$, respectively (see Figure 2 (a) and (c)). In this setting, the sampled initial phases and natural frequencies satisfy

$$r_0 \approx 0.5922, \quad \max_j |\Omega_j| \approx 0.0994, \quad D(\Omega) \approx 0.1987 < 2.$$



(a) Initial configuration (b) Positions of θ_i

(c) Variation of order parameter r (d) Variation of diameter

FIGURE 1. Identical oscillators

Note that formulae (3.5) and (4.14) imply

$$r_* \approx 0.2964 \quad r^* = 0.75 \quad \text{and} \quad \varepsilon_2 \approx 0.0819.$$

As $1.7279 \approx 0.55\pi < \frac{\pi}{2} + \varepsilon_2 \approx 1.7801$, the initial configuration satisfies

$$\theta_i \in [-\frac{\pi}{2} - \varepsilon_2, \frac{\pi}{2} + \varepsilon_2] \quad \text{for all } i = 1, \cdots, 200.$$

On the other hand, Figure 2(e) shows that

$$r(t) \geq r_0 \qquad \text{for all } t \geq 0,$$

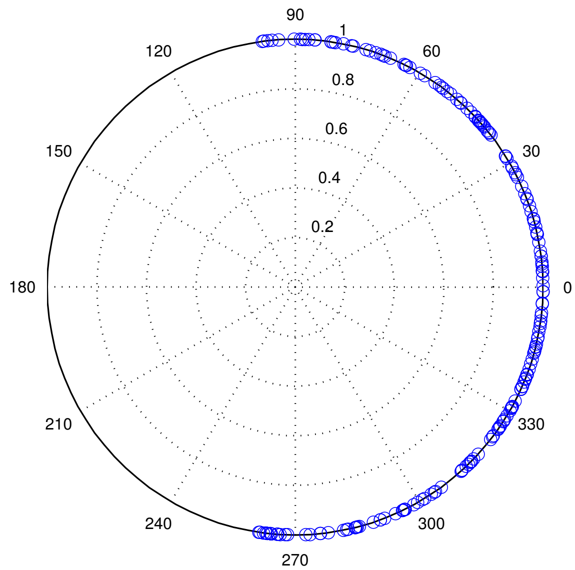
which is consistent with the result in Lemma 3.1. As in the identical oscillator case, the initial phase-diameter is strictly larger than π, i.e., $D(\theta(0)) \approx 3.4216 > \pi$. Thus, the previous result in Theorem 2.1 again fails to explain this case. From (4.23), we get $t_e \approx 0.6554$. Figures 2(d) and 2(f) indicate that $D(\theta(t_e)) \approx 2.1000 < \pi$, which is verified by Lemma 4.3. Therefore, by applying Theorem 4.2, the dynamics of this system can be explained.

6. Conclusion

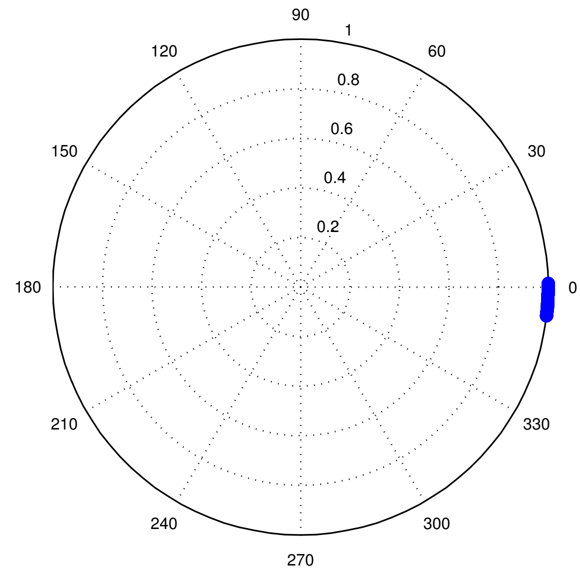
We have presented a new complete frequency synchronization estimate that extends the earlier results in [7, 8, 13, 16]. As discussed in the Introduction, numerical simulations for the Kuramoto oscillators yield complete frequency synchronization for arbitrary initial configurations as long as the coupling strength is sufficiently large. Unfortunately, the analysis does not yet confirm these results. Up to now, for an ensemble of nonidentical oscillators, rigorous complete frequency synchronization estimates have only been confirmed for initial configurations whose diameter is strictly smaller than π. This upper bound of π has been a big obstacle in the complete frequency synchronization problem. In the aforementioned literature, the phase-diameter plays a key role as a Lyapunov functional, and complete frequency synchronization was obtained via a Gronwall-type inequality for this phase-diameter. In this approach, we inevitably find that the π upper bound presents a significant difficulty in the analysis. In this paper, we use both the Lyapunov functional approach based on the phase-diameter and Kuramoto order parameters. The synchronization analysis based on Kuramoto order parameters has been extensively used in the mean-field setting ($N \to \infty$) and in a critical coupling regime where the transition from the disordered state to partially ordered state emerges. In this paper, we extend the range of admissible initial configurations beyond the previous bound of π. Thanks to a detailed analysis of the dynamics of Kuramoto order parameters and evolution of the phase-diameter, we can show that initial configurations whose diameter is larger than π can shrink to a configuration whose diameter is less than π in finite time.

References

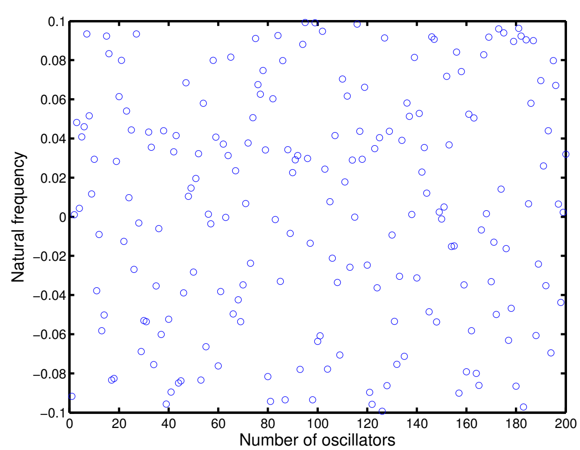
1. Acebron, J. A., Bonilla, L. L., Pérez Vicente, C. J. P., Ritort, F. and Spigler, R.: *The Kuramoto model: A simple paradigm for synchronization phenomena.* Rev. Mod. Phys. **77**, 137-185 (2005).
2. Aeyels, D. and Rogge, J.: *Stability of phase locking and existence of frequency in networks of globally coupled oscillators.* Prog. Theor. Phys. **112**, 921-941 (2004).
3. Bonilla, L. L., Neu, J. C. and Spigler, R.: *Nonlinear stability of incoherence and collective synchronization in a population of coupled oscillators.* J. Stat. Phy. **67**, 313-330 (1992).
4. Balmforth, N. J. and Sassi, R.: *A shocking display of synchrony.* Physica D **143**, 21-55 (2000).
5. Buck, J. and Buck, E.: *Biology of synchronous flashing of fireflies.* Nature **211**, 562 (1966).
6. Chi, D., Choi, S.-H. and Ha, S.-Y.: *Emergent behavior of a holonomic particle system on a sphere.* J. Math. Phys. **55**, 052703 (2014).
7. Choi, Y.-P., Ha, S.-Y., Jung, S. and Kim, Y.: *Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model.* Physica D **241**, 735-754 (2012).
8. Chopra, N. and Spong, M. W.: *On exponential synchronization of Kuramoto oscillators.* IEEE Trans. Automatic Control **54**, 353-357 (2009).
9. Crawford, J. D. and Davies, K. T. R.: *Synchronization of globally coupled phase oscillators: singularities and scaling for general couplings.* Physica D **125**, 1-46 (1999).
10. De Smet, F. and Aeyels, D.: *Partial frequency in finite Kuramoto-Sakaguchi model.* Physica D **234**, 81-89 (2007).
11. Dong, J.-G. and Xue, X.: *Synchronization analysis of Kuramoto oscillators.* Commun. Math. Sci. **11**, 465-480 (2013).
12. Dörfler, F. and Bullo, F.: *Synchronization in complex network of phase oscillators: A survey.* Automatica **50**, 1539-1564 (2014).



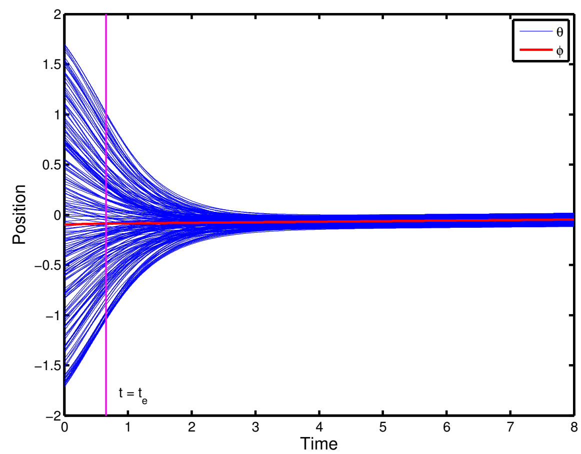
(a) Initial configuration



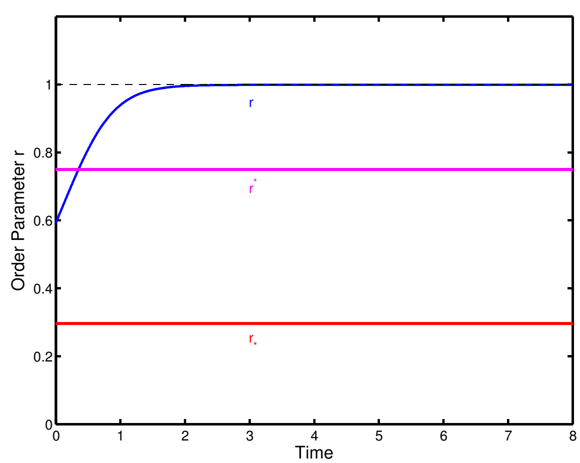
(b) Final configuration



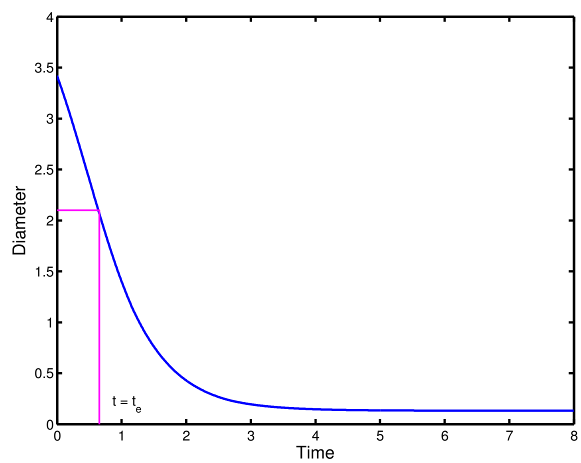
(c) Distribution of natural frequencies



(d) Positions of θ_i



(e) Order parameter r



(f) Diameter

FIGURE 2. Nonidentical oscillators

13. Dörfler, F. and Bullo, F.: *On the critical coupling for Kuramoto oscillators.* SIAM. J. Appl. Dyn. Syst. **10**, 1070-1099 (2011).
14. Ermentrout, G. B.: *Synchronization in a pool of mutually coupled oscillators with random frequencies.* J. Math. Biol. **22**, 1-9 (1985).
15. Evans, Lawrence. Partial Differential Equations. *Graduate Studies in Mathematics*, 19. American Mathematical Society , 1991.
16. Ha, S.-Y., Ha, T. Y. and Kim, J.-H.: *On the complete synchronization for the globally coupled Kuramoto model.* Physica D **239**, 1692-1700 (2010).
17. Ha, S.-Y. and Kang, M.: *Fast and slow relaxations to bi-cluster configuratons for the ensemble of Kuramoto oscillators.* Quart. Appl. Math. **71**, 707-728 (2013).
18. Ha, S.-Y. and Slemrod, M.: *A fast-slow dynamical systems theory for the Kuramoto type phase model.* J. Differential Equations **251**, 2685-2695 (2011).
19. Ha, S.-Y., Li, Z. and Xue, X.: *Formation of phase-locked states in a population of locally interacting Kuramoto oscillators.* J. Differential Equations **255**, 3053-3070 (2013).
20. Jadbabaie, A., Motee, N. and Barahona M.: *On the stability of the Kuramoto model of coupled nonlinear oscillators.* Proceeding of the American Control Conference, 4296-4301 (2004).
21. Kuramoto, Y.: *Chemical Oscillations, waves and turbulence.* Springer-Verlag, Berlin, 1984.
22. Kuramoto, Y.: *International symposium on mathematical problems in mathematical physics.* Lecture Notes in Theoretical Physics. **30**, 420 (1975).
23. Mirollo, R. and Strogatz, S. H.: *The spectrum of the partially locked state for the Kuramoto model.* J. Nonlinear Sci. **17**, 309-347 (2007).
24. Mirollo, R. and Strogatz, S. H.: *The spectrum of the locked states for the Kuramoto model of coupled oscillators.* Physica D **205**, 249-266 (2005).
25. Mirollo, R. and Strogatz, S. H.: *Stability of incoherence in a population of coupled oscillators.* J. Stat. Phys. **63**, 613-635 (1991).
26. Monzón, P. and Paganini, F.: *Global considerations on the Kuramoto model of sinusoidally coupled oscillators.* In: IEEE Conf. on decision and control. San Diego CA, USA Dec. 3923-3928 (2005).
27. Pikovsky, A., Rosenblum, M. and Kurths, J.: *Synchronization: A universal concept in nonlinear sciences.* Cambridge University Press, Cambridge, 2001.
28. Roberts, D. C.: *A linear reformulation of the Kuramoto model of self-synchronizing coupled oscillators.* Phys. Rev. E **77**, 031114 (2008).
29. Scardovi, L., Sarlette, A. and Sepulchre, R.: *Synchronization and balancing on the N-torus.* Systems and Control letters. **56**, 335-341 (2007).
30. Sepulchre, R., Paley, D. A. and Leonard, N. E.: *Stabilization of planar collective motion: all-to-all communication.* IEEE Transactions on Automatic Control **52**, 811-824 (2007).
31. Strogatz, S. H.: *From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators.* Physica D **143**, 1 (2000).
32. van Hemmen, J. L. and Wreszinski, W. F.: *Lyapunov function for the Kuramoto model of nonlinearly coupled oscillators.* J. Stat. Phys. **72**, 145-166 (1993).
33. Verwoerd, M. and Mason, O.: *A convergence result for the Kuramoto model with all-to-all coupling.* SIAM J. Appl. Dyn. Syst. **10**, 906-920 (2011).
34. Verwoerd, M. and Mason, O.: *Global phase-locking in finite populations of phase-coupled oscillators.* SIAM J. Appl. Dyn. Syst. **7**, 134-160 (2008).
35. Winfree, A. T.: *Biological rhythms and the behavior of populations of coupled oscillators.* J. Theor. Biol. **16**, 15-42 (1967).
36. Zhu, J.: *Synchronization of Kuramoto model in a high-dimensional linear space.* Phys. Letter A **377**, 2939-2943 (2013).

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