EMERGENT DYNAMICS OF WINFREE OSCILLATORS ON LOCALLY COUPLED NETWORKS

SEUNG-YEAL HA, DONGNAM KO, JINYEONG PARK, AND SANG WOO RYOO

Abstract. The Winfree model is the first mathematical model for synchronization of weakly coupled oscillators. Compared to the well-known Kuramoto model, the Winfree model does not conserve the total phase. This leads to rich dynamic features compared to those produced by other phase models. In this paper, we study the emergent dynamics of the Winfree model on a locally coupled static network. A randomly chosen phase configuration undergoes several dynamic phase transitions such as incoherence, partial locking, complete locking, partial oscillator death, and complete oscillator death, as the coupling strength increases. We provide several rigorous analytical results on the emergence of these dynamic features. We also provide several numerical simulations and compare their results to the analytical results.

1. Introduction

Collective motions of interacting particle systems are ubiquitous phenomena often observed in complex systems, e.g., the aggregation of bacteria, flocking of birds, swarming of fish, herding of sheep, and flashing of fireflies [3, 4, 8, 10, 11, 12, 15, 20, 22, 29, 39, 41, 42, 43, 44]. Recently, such collective dynamics have received considerable attention from many scientific disciplines because of their diverse engineering applications to the decentralized control of unmanned aerial vehicles [5, 6, 16, 23, 27, 33, 32, 31, 36, 37, 38, 40]. In this paper, we primarily focus on synchronization. Synchronization is the representation of emerging rhythms in oscillatory systems. First discovered by Huygens in a two pendulum clock hanging on the same bar, its rigorous mathematical formulation is fairly recent; in particular, Winfree [44] and Kuramoto [25, 26] formalized the notion of synchronization fifty years ago. Since then, several phase and pulse-coupled models have been proposed and extensively studied both analytically and numerically in the literature. Of these studies, we are interested in the first phase-coupled model for synchronization, namely, the Winfree model. The Winfree model was first proposed in Arthur Winfree's senior thesis [44].

Next, we briefly introduce the Winfree phase model. Weakly coupled phase oscillators can be visualized as rotors moving on the unit circle $\mathbb{S}^1$. In this simple representation, the spatial position of a rotor is determined by its polar angle (phase) using polar coordinates. Let $\theta_i = \theta_i(t)$ be the phase of the i-th oscillator. Then the phase dynamics of Winfree oscillators are

Date: November 27, 2015.

1991 *Mathematics Subject Classification.*

Key words and phrases. Oscillator death, phase model, phase-locked state, synchronization, Winfree model.

Acknowledgment: The work of S.-Y. Ha is supported by a NRF grant (2011-0015388) and the work of J. Park was supported by NRF(National Research Foundation of Korea) Grant funded by the Korean Government(NRF-2014-Fostering Core Leaders of the Future Basic Science Program).

governed by the Cauchy problem for the following first-order system of ordinary differential equations (ODEs):

$$\begin{cases} \dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} c_{ji} S(\theta_i) I(\theta_j), \quad t > 0, \ i = 1, \ldots, N, \\ \theta_i(0) = \theta_{i0}, \end{cases} \tag{1.1}$$

where K, ω_i, and c_{ji} are the positive coupling strength, natural frequency of the i-th oscillator, and communication capacity between the i-th and j-th oscillators, respectively. The coupling functions S and I measure the sensitivity and influences of oscillators. Throughout this paper, we assume that the connection topology $\mathcal{C} = (c_{ij})$ satisfies symmetricity and connectedness:

$$\begin{aligned} & c_{ji} \geq 0, \qquad c_{ji} = c_{ij}, \quad 1 \leq i, j \leq N, \\ & \text{for any } \ i, j \in \{1, \cdots, N\}, \ \text{ there exists a path between } \ i \ \text{ and } \ j, \ \text{ i. e.,} \\ & \exists i = k_0 \to k_1 \to \cdots \to k_{m_{(i,j)}} = j \ \text{ such that } \ c_{k_l, k_{l+1}} > 0, \quad l = 0, \cdots, k_{m_{(i,j)}-1}. \end{aligned} \tag{1.2}$$

The Winfree model in (1.1) with the special pair $(S(\theta), I(\theta)) = (-\sin\theta, 1 + \cos\theta)$ on the all-to-all network $c_{ji} = \frac{1}{N}$ has been studied in literature [2, 28, 30, 34, 35]. However, compared to the extensive work [1, 7, 9, 13, 14, 17, 18, 19] on the Kuramoto model, fewer research exists on the Winfree model (1.1) because of the lack of conservation of total phase and translational symmetry. Moreover, the aforementioned literature on (1.1) mostly deals with the mean-field situation, i.e., an ensemble of infinite Winfree oscillators. To the best of our knowledge, the first analytical study on the Winfree model with a finite size was studied in [21]. For distributed natural frequencies, four dynamic phase states can emerge, as the coupling strength increases from a small value to infinity. More precisely, for a given random initial phase configuration that is in the incoherent state, the following four dynamic phases might emerge as the coupling strength increases:

$$\begin{aligned} &\text{Incoherent state} \quad \Longrightarrow \quad \text{Partial locking} \quad \Longrightarrow \quad \text{Complete locking} \\ &\qquad \Longrightarrow \quad \text{Partial oscillator death} \quad \Longrightarrow \quad \text{Complete oscillator death.} \end{aligned}$$

Of course, some of dynamic phase might not emerge depending on the relative sizes between the coupling strength and natural frequencies. In [2], the diverse phase states of the discrete Winfree oscillators are studied numerically. In the sequel, the dynamics for Winfree models with large number of oscillators are considered in [34]

The main purpose of this paper is to provide a rigorous framework for the emergence of the above dynamic phases in the Winfree model on general symmetric networks. Note that the emergence of complete oscillator death in Winfree oscillators has already been studied [21]; however, only the all-to-all network was considered.

The main results of this paper are as follows. First, we present a sufficient condition leading to partial and complete oscillator deaths in terms of coupling strength, sensitivity, and influence functions (Theorem 4.1). Second, we prove the existence of an attractor with a positive Lebesgue measure that absorbs all neighboring configurations in a large-coupling regime (Proposition 5.1). Third, we provide the exponential ℓ^1-stability (Proposition 5.2), and fourth, we present a sufficient framework for the emergence of a unique equilibrium

(Theorem 5.1).

The rest of this paper is divided into six sections. In Section 2, we review the Winfree model on a network and discuss the hidden coupling mechanism in (1.1). We also briefly discuss the result in [21] for the emergence of complete oscillator death. In Section 3, we present a subclass of Winfree models that can be reformulated as a gradient flow on the symmetric network and present a sufficient framework leading to complete oscillator death. In Section 4, we present a sufficient framework for partial and complete oscillator deaths in terms of coupling strength and coupling functions. In Section 5, we discuss three qualitative properties of the Winfree model; specifically, we discuss the existence of an attractor, exponential ℓ^1-stability, and the unique existence of an equilibrium inside the attractor. In Section 6, we present several numerical simulations and compare them to the analytical results in Sections 3, 4, and 5. Finally, in Section 7, we briefly summarize our main results.

Notation: For $A = (a_1, \cdots, a_N) \in \mathbb{R}^N$ and $N \in \mathbb{Z}_+$, we set

$$\|A\|_p := \Big(\sum_{i=1}^N |a_i|^p \Big)^{\frac{1}{p}}, \quad 1 \le p < \infty, \qquad \|A\|_\infty := \max_{1 \le i \le N} |a_i|, \quad \mathcal{N} := \{1, \cdots, N\}.$$

2. Preliminaries

In this section, we briefly discuss Winfree's idea for synchronization modeling of the ensemble of weakly coupled limit-cycle oscillators in (1.1). We also review a previous result in [21] on the emergence of complete oscillator death in the Winfree model.

First, recall several concepts of dynamic phases in relation to the collective dynamics of (1.1) in terms of the rotation number $\rho := \lim_{t \to \infty} \dfrac{\theta(t)}{t}$ (see [24]).

Definition 2.1. *Let $\Theta := (\theta_1, \cdots, \theta_N)$ be an ensemble of Winfree oscillators whose dynamics are governed by* (1.1).

(1) *The configuration Θ tends to "complete oscillator death (COD)" if and only if the rotation numbers of all oscillators are zero, i.e.,*

$$|\{i : \rho_i = 0\}| = N,$$

where $|A|$ is the size of set A.

(2) *The configuration Θ tends to "partial oscillator death (POD)" if and only if the rotation numbers of at least two oscillators are zero. i.e.,*

$$2 \le |\{i : \rho_i = 0\}| < N.$$

Note that for POD, not all rotation numbers are zero; if all oscillator rotation numbers are zero, COD is achieved.

(3) *The configuration Θ tends to "complete phase-locked state (CPLS)" if and only if the rotation numbers of all oscillators are equal and nonzero, i.e., there exists a nonzero number ρ such that*

$$|\{i : \rho_i = \rho\}| = N.$$

(4) *The configuration Θ tends to "partially phase-locked state (PPLS)" if and only if there exist at least two oscillators whose rotation numbers are the same, i.e., there exists a nonzero constant ρ such that*

$$2 \le |\{i \ : \ \rho_i = \rho\}| < N.$$

Note that for PPLS, not all rotation numbers are the same; if all oscillator rotation numbers are equal to ρ, CPLS is achieved.

Remark 2.1. *For bounded distributed natural frequencies, the phase diagram illustrating the transition from incoherence to oscillator death in the mean-field setting, using linear analysis, can be found in* [2].

2.1. **Network Winfree models.** Consider a static symmetric network modeled by the weighted graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{C})$. Assume that Winfree oscillators are located on the vertices of the network $\mathcal{G}$ and interact via a connectivity matrix registered by $(\mathcal{E}, \mathcal{C})$. In 1967, Winfree [44] proposed a phase model for the synchronization of weakly coupled limit-cycle oscillators on the all-to-all network:

$$\mathcal{E} = \mathcal{N} \times \mathcal{N}, \qquad \mathcal{C} = \frac{1}{N}\mathbf{1},$$

where $\mathbf{1}$ is the constant matrix whose elements are unity, i.e., $c_{ij} = \frac{1}{N}$. To better visualize this situation, we can view limit-cycle oscillators as point rotors moving on the unit circle $\mathbb{S}^1$. Let $\theta_i = \theta_i(t)$ and ω_i be the phase and natural frequency, respectively, of the i-th oscillator. When there are no mutual interactions between oscillators, the dynamics of the oscillator phase is completely determined by the natural frequency of the oscillator:

$$\dot{\theta}_i = \omega_i, \quad \text{i.e.,} \quad \theta_i(t) = \theta_{i0} + \omega_i t, \qquad i = 1, \cdots, N. \tag{2.3}$$

Thus, the difference between the rotation numbers of oscillators is simply the differences between natural frequencies:

$$\rho_i - \rho_j = \lim_{t\to\infty} \frac{\theta_i(t) - \theta_j(t)}{t} = \omega_i - \omega_j.$$

Note that for $\omega_i \ne \omega_j$, the rotation numbers ρ_i and ρ_j cannot be equal, i.e., entrainment between the i-th and j-th oscillators does not occur. Thus, the question then arises: how should coupling be introduced between oscillators to make them entrained? To answer this question, we review Winfree's seminal idea. In the presence of mutual interactions, the dynamics in (3.11) should be supplemented by adding the synchronizing forcing terms, $\hat{\omega}_i$ to the right-hand side of (3.11), which registers the weak interactions between oscillators; that is,

$$\dot{\theta}_i = \omega_i + \hat{\omega}_i, \quad i = 1, \cdots, N.$$

Let $I = I(\theta)$ and $S = S(\theta)$ be the influence and sensitivity functions, respectively. More precisely, $I(\theta)$ represents the quantifiable measure of influence on neighboring oscillators when the test oscillator has the phase θ. $S(\theta)$, on the other hand, measures the sensitivity (response) of the stimulus on neighboring oscillators. Winfree's pioneering idea for mutual interactions can be summarized as follows [44].

- $(\mathcal{A}1)$: The stimulus I_c^i on the i-th oscillator is the weighted sum of neighboring oscillators' influences, i.e.,

$$I_c^i(\Theta) := \sum_{j=1}^N c_{ji} I(\theta_j). \tag{2.4}$$

  In this case, two assumptions are made in (2.4). First, the influences of oscillators are assumed to be propagated *without attenuation* in time and *time-delay*, in a time much shorter than the average period of the oscillators. Second, they are additive in their effects.

- $(\mathcal{A}2)$: The frequency perturbation $\hat{\omega}_i$ is proportional to the product of the sensitivity $S(\theta_i)$ and the weighted sum stimulus $I_c(\Theta)$:

$$\begin{aligned}\hat{\omega}_i &= K \underbrace{S(\theta_i)}_{\text{response of } i\text{-th oscillator}} \times \underbrace{I_c^i(\Theta)}_{\text{total stimulus by neighboring oscillators}} \\ &= K S(\theta_i) \sum_{j=1}^N c_{ji} I(\theta_j),\end{aligned}$$

  where K is the uniform proportional constant, i.e., the uniform coupling strength.

Based on postulates $(\mathcal{A}1)$ and $(\mathcal{A}2)$, the Winfree model on a network $\mathcal{G}$ is as follows:

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^N c_{ji} S(\theta_i) I(\theta_j), \quad i \in \mathcal{N}.$$

2.2. **Emergence of COD in all-to-all networks.** Consider the Winfree model in (1.1) on an all-to-all network with $c_{ji} = \frac{1}{N}$:

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^N S(\theta_i) I(\theta_j), \quad i \in \mathcal{N}, \quad i \in \mathcal{N},\ t > 0. \tag{2.5}$$

In [21], a sufficient framework leading to COD is presented. Again, let S and I be the sensitivity and influence functions, respectively.

- $(\mathcal{F}1)$: The sensitivity function S is a 2π-periodic, $\mathcal{C}^2$-odd function, and the influence function I is a 2π-periodic $\mathcal{C}^2$-even function; more precisely,

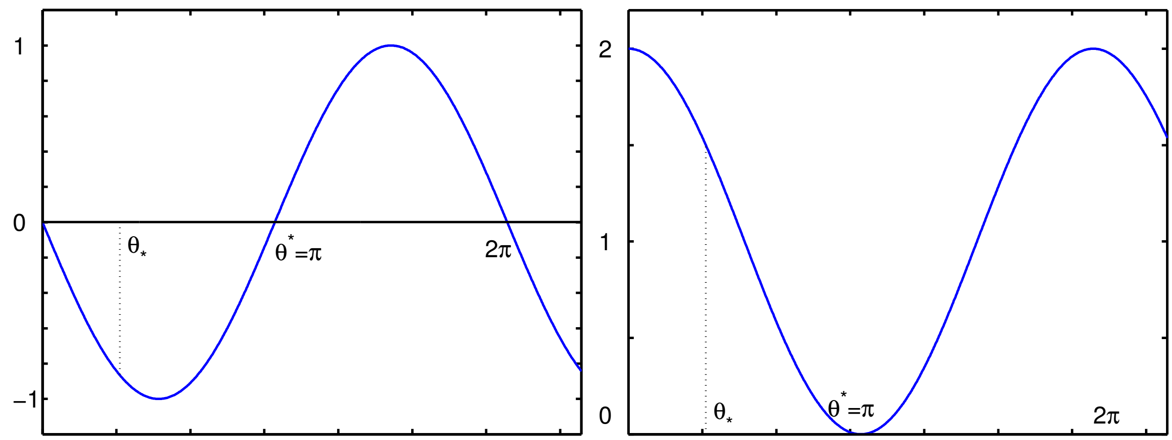
$$\begin{aligned}&S \in \mathcal{C}^2(\mathbb{R}), \quad S(\theta + 2\pi) = S(\theta), \quad S(-\theta) = -S(\theta), \quad \theta \in \mathbb{R}, \\ &I \in \mathcal{C}^2(\mathbb{R}), \quad I(\theta + 2\pi) = I(\theta), \quad I(-\theta) = I(\theta).\end{aligned} \tag{2.6}$$

- $(\mathcal{F}2)$: The sensitivity and influence functions satisfy some geometric conditions; specifically, there exist positive constants θ_* and θ^*, satisfying

$$0 < \theta_* < \theta^* < 2\pi,$$

  such that,

$$\begin{aligned}&S \le 0 \quad \text{on } [0, \theta^*] \quad \text{and} \quad S' \le 0, \quad S'' \ge 0 \quad \text{on } [0, \theta_*], \\ &I \ge 0, \quad I' \le 0 \quad \text{on } [0, \theta^*], \quad \text{and} \quad I'' \le 0 \quad \text{on } [0, \theta_*], \\ &(SI)' < 0 \quad \text{on } (0, \theta_*), \quad (SI)' > 0 \quad \text{on } (\theta_*, \theta^*),\end{aligned} \tag{2.7}$$



(a) $S(\theta) = -\sin\theta$ (b) $I(\theta) = 1 + \cos\theta$

FIGURE 1. Schematic diagrams for $S(\theta)$ and $I(\theta)$.

where S' denotes the θ-derivative of S (see Figure 1 for schematic graphs of S and I).

Note that $S(0) = 0$, and the following special pair (S, I), employed in [2, 30, 34, 35] satisfy the structural conditions in (2.6) and (2.7):

$$S(\theta) = -\sin\theta, \qquad I(\theta) = 1 + \cos\theta, \qquad \theta_* = \frac{\pi}{3}, \qquad \theta^* = \pi. \tag{2.8}$$

Before we describe our main result, we first introduce some notation. For a given $\alpha \in (0, \theta^*)$, consider the following equation on the interval $[0, \theta_*]$:

$$(SI)(x) = (SI)(\alpha), \quad x \in [0, \theta_*]. \tag{2.9}$$

Note that conditions (2.6) and (2.7) yield the following geometric shape of the coupling function SI (see Figure 2):

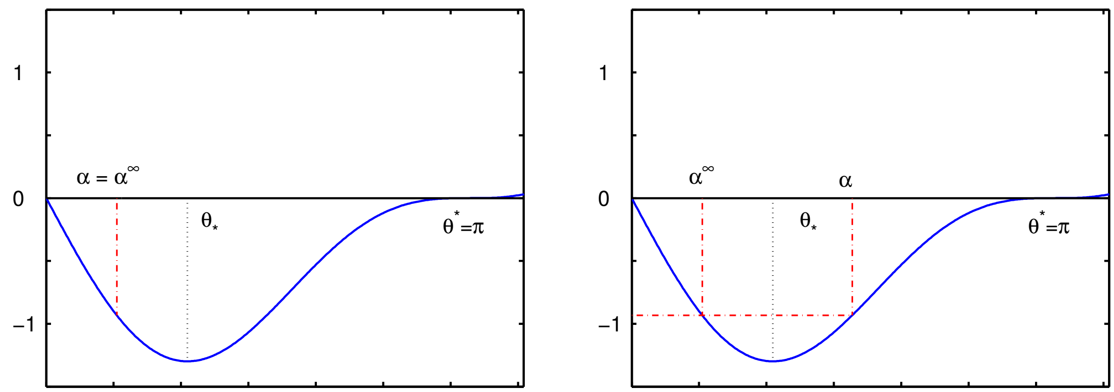
$$\begin{aligned} &(SI)(0) = 0, \quad \theta_* = \mathrm{argmin}_{0\le\theta\le\theta^*}(SI)(\theta), \\ &(SI)(\theta) < 0 \quad \text{on } \theta \in (0, \theta^*), \quad (SI)(\theta^*) \le 0. \end{aligned} \tag{2.10}$$

Thus, (2.9) has a unique solution α^∞, guaranteed by (2.10). There is $\alpha^\infty \in (0, \theta_*)$ satisfying $SI(\alpha^\infty) = SI(\alpha)$ for $\alpha \in (\theta_*, \theta^*)$ as in Figure 2(b). For $\alpha \in (0, \theta_*]$, we have $\alpha^\infty = \alpha$ as in Figure 2(a). For such α^∞, we define the coupling strength $K_e(\alpha^\infty)$ and a set $\mathcal{R}(\alpha^\infty)$ as follows:

$$\begin{aligned} &K_e(\alpha^\infty) := -\frac{\max_i |\omega_i|}{S(\alpha^\infty) I(\alpha^\infty)} \quad \text{and} \\ &\mathcal{R}(\alpha^\infty) := \{\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N \mid \theta_i \in (-\alpha^\infty, \alpha^\infty),\ i = 1, \dots, N\}. \end{aligned}$$

Theorem 2.1. [21] *Suppose conditions* (2.6) *and* (2.7) *hold. For* $\alpha \in (0, \theta^*)$, *let* $\Theta = \Theta(t)$ *be a global smooth solution to system* (2.5), *satisfying*

$$\Theta_0 \in \overline{\mathcal{R}(\alpha)} \quad \textit{and} \quad K > K_e(\alpha^\infty).$$



(a) Position of α^∞ for $\alpha \in (0, \theta_*)$. (b) Position of α^∞ for $\alpha \in (\theta_*, \theta^*)$.

FIGURE 2. Relation between α and α^∞.

Then $\Theta(t)$ converges to the unique equilibrium state $\Phi = (\phi_1, \cdots, \phi_N)$ in the region $\mathcal{R}(\alpha^\infty)$, i.e., there exists a unique phase-locked state $\Phi := (\phi_1, \dots, \phi_N) \in \mathcal{R}(\alpha^\infty)$, such that

$$\omega_i + \frac{K}{N} S(\phi_i) \sum_{j=1}^N I(\phi_j) = 0, \qquad \lim_{t\to\infty} \Theta(t) = \Phi.$$

Remark 2.2. *The result of Theorem 2.1 implies the emergence of COD as follows. Let ρ_i be the rotation number of the i-th oscillator. It follows from Theorem 2.1 that*

$$\theta_i(t) = \phi_i.$$

Thus, the rotation number ρ_i is zero:

$$\rho_i = \lim_{t\to\infty} \frac{\theta_i(t)}{t} = 0.$$

In the following two sections, we study the emergent dynamics for gradient and general systems in relation to (1.1).

3. A GRADIENT FLOW FORMULATION OF THE WINFREE MODEL

In this section, we present emergent dynamics of the Winfree model in (1.1) on a symmetric network with the following special relation between S and I:

$$c_{ji} = c_{ij}, \quad 1 \le i, j \le N, \qquad S(\theta) = I'(\theta), \qquad I: \text{ analytic}. \tag{3.11}$$

Note that the well-studied example in (2.8) satisfies (3.11). Together, systems (1.1)-(3.11) become

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^N c_{ji} I'(\theta_i) I(\theta_j), \quad i \in \mathcal{N}. \tag{3.12}$$

Next, we define an analytical potential function $V = V(\Theta)$:

$$V(\Theta) := -\sum_{i=1}^N \omega_i \theta_i - \frac{K}{2} \left(\sum_{i,j=1}^N c_{ji} I(\theta_i) I(\theta_j) \right).$$

It is easy to verify that (3.12) can be rewritten as a gradient system:

$$\dot{\theta}_i = -\partial_{\theta_i} V(\Theta), \qquad \text{i.e.,} \qquad \dot{\Theta} = -\nabla_\Theta V(\Theta).$$

Next, we study the emergence of COD in a large coupling regime. For a gradient system with an analytical potential function, uniform boundedness implies convergence toward the equilibrium state. Thus, we have the emergence of COD.

Proposition 3.1. [19]. *Let $\Theta = \Theta(t)$ be a uniformly bounded solution to (3.12) satisfying*

$$\sup_{t>0} \|\Theta(t)\|_\infty < \infty.$$

Then there exists an equilibrium Θ^∞ such that

$$\lim_{t\to\infty} \|\Theta(t) - \Theta^\infty\|_\infty = 0. \tag{3.13}$$

Remark 3.1. *Note that the convergence relation in (3.13) causes the rotation numbers ρ_i to approach zero. Thus, Proposition 3.1 implies the emergence of COD.*

We set Ω and d_i to be the natural frequency vector and degree of the i-th node, respectively:

$$\Omega := (\omega_1, \cdots, \omega_N), \qquad d_i := \sum_{j=1}^N c_{ji}.$$

Lemma 3.1. *Suppose that the functions S and I satisfy (2.7) and (3.11), and let $\Theta = \Theta(t)$ be the solution to (3.12) satisfying*

$$\theta_{i0} \in [-\theta_*, \theta_*],\ i \in \mathcal{N} \quad \textit{and} \quad K > \left(\frac{\|\Omega\|_\infty}{\min\limits_{1\le i\le N} d_i}\right)\frac{1}{|I'(\theta_*)I(\theta_*)|}.$$

Then, we have

$$\theta_i(t) \in [-\theta_*, \theta_*], \quad i \in \mathcal{N} \quad \textit{for} \quad t > 0.$$

Proof. We will show that the interval $\mathcal{I} := [-\theta_*, \theta_*]$ is a positively invariant set. To this end, it suffices to show that once the flow hits the boundary points $-\theta_*$ or θ_* at some finite time t_0, then it will flow into the interval $\mathcal{I}$ again so that the flow is confined in the closed interval $\mathcal{I}$ afterwards. Since (3.12) is autonomous, without loss of generality, we assume $t_0 = 0$.

- Case A: Suppose that there exists $i \in \mathcal{N}$ such that

$$\theta_{i0} = -\theta_* \quad \text{and} \quad \theta_{j0} \in [-\theta_*, \theta_*], \quad 1 \le j \ne i \le N.$$

In this case, we use the facts that

$$\begin{aligned} &\omega_i \ge -\|\Omega\|_\infty, \quad I'(\theta_{i0}) = I'(-\theta_*) = -I'(\theta_*) > 0 \quad \text{and} \\ &I(\theta_{j0}) \ge I(-\theta_*) = I(\theta_*) > 0 \end{aligned} \tag{3.14}$$

to show

$$
\begin{aligned}
\frac{d\theta_i}{dt}\Big|_{t=0+} &= \omega_i + KI'(\theta_{i0})\sum_{j=1}^N c_{ji}I(\theta_{j0}) \\
&\geq \omega_i - KI'(\theta_*)\sum_{j=1}^N c_{ji}I(\theta_*) \qquad \text{by (3.14)} \\
&= \omega_i + Kd_i|I'(\theta_*)I(\theta_*)| \qquad \text{using } I'(\theta_*)I(\theta_*) < 0 \\
&\geq -\|\Omega\|_\infty + K\Big(\min_{1\leq i\leq N} d_i\Big)|I'(\theta_*)I(\theta_*)| \qquad \text{by (3.14)} \\
&> 0.
\end{aligned}
\tag{3.15}
$$

• Case B: Suppose that there exists $i \in \mathcal{N}$ such that

$$\theta_{i0} = \theta_* \quad \text{and} \quad \theta_{j0} \in [-\theta_*, \theta_*], \quad 1 \leq j \neq i \leq N.$$

Then,

$$\omega_i \leq \|\Omega\|_\infty, \quad I'(\theta_{i0}) = I'(\theta_*) < 0, \quad I(\theta_{j0}) \geq I(\theta_*) > 0, \tag{3.16}$$

implies

$$
\begin{aligned}
\frac{d\theta_i}{dt}\Big|_{t=0+} &= \omega_i + K\sum_{j=1}^N c_{ji}I'(\theta_{i0})I(\theta_{j0}) \\
&\leq \omega_i + K\sum_{j=1}^N c_{ji}I'(\theta_*)I(\theta_*) \qquad \text{by (3.16)} \\
&= \omega_i - Kd_i|I'(\theta_*)I(\theta_*)| \qquad \text{using } I'(\theta_*)I(\theta_*) < 0 \\
&\leq \|\Omega\|_\infty - K(\min_{1\leq i\leq N} d_i)|I'(\theta_*)I(\theta_*)| \qquad \text{by (3.16)} \\
&< 0.
\end{aligned}
\tag{3.17}
$$

Finally, we combine the results in (3.15) and (3.17) of Cases A and B to conclude $\mathcal{I}$ is a positively invariant set. □

By combining the uniform boundedness of $\Theta(t)$ and Proposition 3.1, we derive the following result.

Theorem 3.1. *Suppose that the functions S and I satisfy* (5.35) *and the structural condition in* (2.7)*, and let $\Theta = \Theta(t)$ be the solution to* (3.12) *satisfying*

$$\theta_{i0} \in [-\theta_*, \theta_*],\ i \in \mathcal{N} \quad \text{and} \quad K > \Big(\frac{\|\Omega\|_\infty}{\min_{1\leq i\leq N} d_i}\Big)\frac{1}{|I'(\theta_*)I(\theta_*)|}.$$

Then COD occurs, i.e.,

$$\rho_i = 0, \quad i \in \mathcal{N}.$$

Proof. By Lemma 3.1, the configuration Θ is uniformly bounded, i.e.,

$$\theta_i(t) \in [-\theta_*, \theta_*], \quad i \in \mathcal{N}.$$

It follows from Proposition 3.1 that there exists an equilibrium Θ_e such that

$$\lim_{t\to\infty} \Theta(t) = \Theta_e.$$

This yields

$$\rho_i = \lim_{t\to\infty} \frac{\theta_i(t)}{t} = 0.$$

□

4. A Winfree model on general connected networks

In this section, we study the emergence of COD and POD in the locally coupled Winfree model (1.1)-(1.2) which may not be a gradient system. As in Section 3, it suffices to show that the Winfree flow is uniformly bounded to derive COD in the sense of Definition 2.1 in a large coupling regime.

For $\alpha \in (0, \theta^*)$, we set $\alpha^\infty \in (0, \theta_*)$ to be a mirror point of α determined by relations (2.9) and (2.10). Then, for such α^∞, we set

$$\mathcal{J}(\alpha^\infty) := (-\alpha^\infty, \alpha^\infty).$$

In the future, we plan to show that the symmetric closed interval $\overline{\mathcal{J}}(\alpha^\infty)$ is an attractor for the Winfree flow in (1.1)-(1.2).

Lemma 4.1. (Positive invariance of $\overline{\mathcal{J}}(\alpha^\infty)$) *Suppose that conditions* (2.6)-(2.7) *and* $0 < \alpha < \theta^*$ *hold, and let* $\Theta = \Theta(t)$ *be the global smooth solution to* (1.1)-(1.2) *satisfying*

$$\theta_{i0} \in \overline{\mathcal{J}}(\alpha^\infty),\ i \in \mathcal{N} \quad \text{and} \quad K > \frac{\|\Omega\|_\infty}{(\min_{1\le i\le N} d_i)|(SI)(\alpha^\infty)|}.$$

Then $\overline{\mathcal{J}}(\alpha^\infty)$ *is positively invariant along the Winfree flow* (1.1)-(1.2), *i.e.,*

$$\theta_i(t) \in \overline{\mathcal{J}}(\alpha^\infty), \quad t \ge 0.$$

Proof. We use essentially the same arguments as Lemma 3.1 to show that if the Winfree flow issued from the interval $\overline{\mathcal{J}}(\alpha^\infty)$ hits the boundary of $\mathcal{J}(\alpha^\infty)$, it directs toward the interior of $\mathcal{J}(\alpha^\infty)$ so that $\overline{\mathcal{J}}(\alpha^\infty)$ is a positively invariant region. More precisely, we consider the following two cases.

• Case A: If there exist $t_0 \ge 0$ and $i \in \mathcal{N}$ such that

$$\theta_i(t_0) = -\alpha^\infty, \quad \theta_j(t_0) \in \overline{\mathcal{J}}(\alpha^\infty),$$

then, by the properties of S,

$$S(\theta_i(t_0)) = S(-\alpha^\infty) = |S(\alpha^\infty)|, \quad I(\theta_j(t_0)) \ge I(\alpha^\infty) > 0.$$

This yields

$$\begin{aligned} \frac{d\theta_i}{dt}\Big|_{t=t_0} &\ge -\|\Omega\|_\infty + K\sum_{j=1}^N c_{ji} S(\theta_i(t_0)) I(\theta_j(t_0)) \\ &\ge -\|\Omega\|_\infty + K\sum_{j=1}^N c_{ji} |(SI)(\alpha^\infty)| \qquad (4.18) \\ &\ge -\|\Omega\|_\infty + K(\min_{1\le i\le N} d_i)|(SI)(\alpha^\infty)| \\ &> 0. \end{aligned}$$

Thus, once θ_i hits the left boundary of $\mathcal{J}(\alpha^\infty)$ in finite time, it will flow into the interval $\mathcal{J}(\alpha^\infty)$, i.e., it cannot leave the interval through the left end point of the interval.

• Case B: If there exist $t_0 \geq 0$ and $i \in \mathcal{N}$ such that

$$\theta_i(t_0) = \alpha^\infty, \quad \theta_j(t_0) \in \overline{\mathcal{J}}(\alpha^\infty),$$

then the properties of S imply

$$S(\theta_i(t_0)) = S(\alpha^\infty) = -|S(\alpha^\infty)|, \quad I(\theta_j(t_0)) \geq I(\alpha^\infty) > 0.$$

This yields

$$\begin{aligned}
\frac{d\theta_i}{dt}\Big|_{t=t_0} &\leq \|\Omega\|_\infty + K\sum_{j=1}^N c_{ji} S(\theta_i(t_0)) I(\theta_j(t_0)) \\
&\leq \|\Omega\|_\infty - K\sum_{j=1}^N c_{ji} |(SI)(\alpha^\infty)| \\
&\leq \|\Omega\|_\infty - K(\min_{1\leq i\leq N} d_i)|(SI)(\alpha^\infty)| \\
&< 0.
\end{aligned} \tag{4.19}$$

Thus, once θ_i hits the right boundary of $\mathcal{J}(\alpha^\infty)$ in finite time, it flows into the interval $\mathcal{J}(\alpha^\infty)$ again, i.e., it cannot leave the interval through the right end point of the interval. Finally, we combine the estimates in (4.18) and (4.19) to obtain the desired result. □

Remark 4.1. *In* (4.18) *and* (4.19), *we only used the non-negativity of* c_{ji} *and positive degree* $d_i > 0$. *We also did not use the symmetric assumption* $c_{ji} = c_{ij}$.

Lemma 4.2. *Suppose that conditions* (2.6) *and* (2.7) *hold, and let* $\Theta = \Theta(t)$ *be the global smooth solution to* (1.1)-(1.2) *satisfying*

$$\alpha \in (0, \theta^*), \quad \theta_{i0} \in \overline{\mathcal{J}}(\alpha),\ i \in \{1, \cdots, n\}, \quad \textit{and} \quad K \geq \max_{1\leq i\leq n} \frac{|\omega_i|}{\sum_{j=1}^n c_{ji}} \frac{1}{|(SI)(\alpha^\infty)|},$$

for some $n \leq N$. *Then, we have*

$$|\theta_i(t)| \leq \alpha, \quad t \geq 0, \quad i \in \{1, \cdots, n\}.$$

Proof. Similar to the proof of Lemma 4.1, we show that the flow of $\{\theta_i\}_{i=1}^n$ cannot leave the closed cube $(\overline{\mathcal{J}}(\alpha))^n$ in a finite time. Since the flow is autonomous, it suffices to show that once the flow initially hits the boundary of the cube $(\overline{\mathcal{J}}(\alpha))^n$, it cannot leave the cube afterwards.

Suppose that $|\theta_{i0}| = \alpha$. Then, we use

$$\mathrm{sgn}(\theta_{i0}) S(\theta_{i0}) = S(|\theta_{i0}|) = S(\alpha) < 0 \quad \text{and} \quad I(\theta_i) \geq I(\alpha), \quad \text{for} \ \ i \in \{1, \cdots, n\}$$

to obtain

$$
\begin{aligned}
\frac{d|\theta_i|}{dt}\Big|_{t=0+} &\le |\omega_i| + K\sum_{j=1}^{N} c_{ji} S(|\theta_{i0}|) I(\theta_{j0}) \\
&\le |\omega_i| + K\sum_{j=1}^{n} c_{ji} S(\alpha) I(\alpha) \\
&= |\omega_i| - K\sum_{j=1}^{n} c_{ji} |(SI)(\alpha^\infty)| \\
&\le 0,
\end{aligned} \tag{4.20}
$$

where the following property of SI is used:

$$\max_{0\le\theta\le\theta^*} (SI)(\theta) \le 0, \quad (SI)(\alpha^\infty) = (SI)(\alpha).$$

□

Remark 4.2. *In Lemma 4.2, we present a positive invariance for some part of oscillators* $\{1, \cdots, n\} \subset \mathcal{N}$*, whereas Lemma 4.1 gives a positively invariant set for the whole oscillators* $\mathcal{N}$*. Note that*

$$\frac{|\omega_i|}{d_i} \le \frac{\|\Omega\|_\infty}{\min_{1\le i\le N} d_i}.$$

Thus, the result of Lemma 4.2 implies that if $K \ge \dfrac{\|\Omega\|_\infty}{\min_{1\le i\le N} d_i}\dfrac{1}{|(SI)(\alpha^\infty)|}$*, then*

$$|\theta_{i0}| \le \alpha \quad \Longrightarrow \quad |\theta_i(t)| \le \alpha \quad \textit{for} \quad i \in \mathcal{N}.$$

The direct application of Lemma 4.2 yields a sufficient condition for the zero rotation number.

Theorem 4.1. (Emergence of POD and COD) *Suppose that conditions* (2.6) *and* (2.7) *hold, and let* $\Theta = \Theta(t)$ *be the global smooth solution to* (1.1)-(1.2) *satisfying*

$$\alpha \in (0, \theta^*), \quad \theta_{i0} \in \mathcal{J}(\alpha),\ i \in \{1, \cdots, n\} \quad \textit{and} \quad K \ge \max_{1\le i\le n} \frac{|\omega_i|}{\sum_{j=1}^n c_{ji}} \frac{1}{|(SI)(\alpha^\infty)|},$$

for some $n \le N$*. Then, we have*

$$\rho_i = 0, \quad i \in \{1, \cdots, n\}.$$

Proof. It follows from Lemma 4.2 that

$$\sup_{t\ge0} |\theta_i(t)| \le \alpha.$$

The definition of ρ_i implies

$$|\rho_i| \le \limsup_{t\to\infty} \frac{|\theta_i(t)|}{t} = 0, \quad \rho_i = 0.$$

□

Remark 4.3. *If we take* $n = N$ *and* $K \ge \dfrac{\|\Omega\|_\infty}{\min_{1\le i\le N} d_i}\dfrac{1}{|(SI)(\alpha^\infty)|}$*, then the result of Theorem 4.1 yields the emergence of COD.*

5. Qualitative properties of Winfree oscillators

In this section, we study the following qualitative properties: existence of an attractor, equilibrium, and exponential ℓ^1-stability.

5.1. **Existence of an attractor.** In this subsection, we show that the set $\mathcal{J}(\alpha^\infty)$ absorbs the neighboring flow, i.e., $\mathcal{J}(\alpha^\infty)$ satisfies an attractivity property. Note that Lemma 4.1 implies that the interval $\mathcal{J}(\alpha^\infty)$ is a bounded and positively invariant set. Thus, we need to show that $\mathcal{J}(\alpha^\infty)$ does attract neighboring configurations in finite time.

Proposition 5.1. *Suppose that conditions* (2.6) *and* (2.7) *hold, and let* $\Theta = \Theta(t)$ *be the global smooth solution to* (1.1)-(1.2) *satisfying*

$$\alpha \in (0, \theta^*), \quad \theta_{i0} \in \mathcal{J}(\alpha),\ i \in \mathcal{N} \quad \textit{and} \quad K > \frac{\|\Omega\|_\infty}{\min\limits_{1 \le i \le N} d_i} \frac{1}{|(SI)(\alpha^\infty)|}.$$

Then there exists a $t_e \in [0, \infty)$ *such that*

$$\theta_i(t) \in \mathcal{J}(\alpha^\infty), \quad t \ge t_e.$$

More precisely, for $\alpha > \alpha^\infty$, t_e *can be chosen as follows:*

$$t_e := -\frac{\alpha - \alpha^\infty}{\Delta(\Omega, D)} > 0, \qquad \Delta(\Omega, D) := \|\Omega\|_\infty - K(\min_{1 \le i \le N} d_i)|(SI)(\alpha^\infty)|.$$

Proof. Suppose that the initial data satisfy

$$\alpha \in (0, \theta^*), \quad \theta_{i0} \in \mathcal{J}(\alpha) \quad \text{for all } i \in \mathcal{N}.$$

Without loss of generality, we assume $\alpha > \alpha^\infty$. Next, we show that the N-dimensional cube $(\mathcal{J}(\alpha))^N$ shrinks into the smaller positively invariant set $\mathcal{J}(\alpha^\infty)^N$ in finite time along the Winfree flow (1.1)-(1.2).

If $\Theta_0 \in (\mathcal{J}(\alpha^\infty))^N$, then taking $t_e = 0$ yields the desired result. Thus, it suffices to show that

$$\Theta_0 \in (\mathcal{J}(\alpha))^N \cap \Big((\mathcal{J}(\alpha^\infty))^N \Big)^c, \tag{5.21}$$

i.e.,

$$\exists\ i \in \mathcal{N} \text{ such that } \alpha^\infty \le |\theta_{i0}| < \alpha \quad \text{and} \quad |\theta_{k0}| < \alpha, \quad k \ne i. \tag{5.22}$$

Next, we show that the initial configuration satisfying (5.21) enters the set $\mathcal{J}(\alpha^\infty)$ in finite time. For this, we define a set $\mathcal{T}$ and its supremum:

$$\mathcal{T} := \Big\{ T \in (0, \infty] :\ |\theta_i(t)| < \alpha + t\Delta(\Omega, D),\ t \in [0, T),\ 1 \le i \le N \Big\}, \\ \text{and} \quad T^* := \sup \mathcal{T}.$$

• Step A ($\mathcal{T}$ is nonempty): Consider the following two cases.

⋄ Case A.1: Let $M := \mathrm{argmax}_{1 \le i \le N} |\theta_i|$, then M satisfies (5.22), we use

$$S(|\theta_{M0}|) < 0,\ I(|\theta_{j0}|) > (|\theta_{M0}|) > I(\alpha) > 0,\ (SI)(\alpha) = (SI)(\alpha^\infty), (SI)(|\theta_{M0}|) \le (SI)(\alpha)$$

to obtain

$$\frac{d|\theta_M|}{dt}\Big|_{t=0+} \quad \le \quad |\omega_M| + K \sum_{j=1}^N c_{jM} S(|\theta_{M0}|) I(|\theta_{j0}|)$$

$$
\begin{aligned}
&\le |\omega_M| + K\sum_{j=1}^N c_{jM} S(|\theta_{M0}|) I(|\theta_{j0}|) \\
&\le |\omega_M| + K\sum_{j=1}^N c_{jM} S(\alpha) I(\alpha) \\
&= |\omega_M| - K\sum_{j=1}^N c_{jM} |(SI)(\alpha)| \\
&< |\omega_M| - K d_M |(SI)(\alpha^\infty)| \\
&\le \Delta(\Omega, D) < 0.
\end{aligned}
$$

Thus, there exists $\delta_i > 0$ such that

$$|\theta_i(t)| \le |\theta_{M0}| + t\Delta(\Omega, D) < \alpha + t\Delta(\Omega, D), \quad t \in [0, \delta_i).$$

$\diamond$ Case A.2: If j satisfies $|\theta_{j0}| < \alpha^\infty$, we consider the continuous function

$$h(t) := \alpha + t\Delta(\Omega, D) - |\theta_j(t)|.$$

Note that since $\alpha^\infty < \alpha$, we have

$$h(0) = \alpha - |\theta_{j0}| \ge \alpha - \alpha^\infty = 0.$$

Then, by the continuity of h, there exists $\delta_j > 0$ such that

$$h(t) > 0, \quad \text{i.e.,} \quad |\theta_j(t)| < \alpha + t\Delta(\Omega, D), \quad t \in [0, \delta_j).$$

Therefore, by choosing $\delta := \min_{1 \le i \le N} \delta_i > 0$, we have

$$\max_{1 \le i \le N} |\theta_i(t)| < \alpha + t\Delta(\Omega, D), \quad t \in [0, \delta), \quad \text{i.e.,} \quad \delta \in \mathcal{T}.$$

Hence, the set $\mathcal{T}$ is nonempty. Thus, $T^* = \sup \mathcal{T} > 0$. Moreover, $T^* \in \mathcal{T}$:

$$|\theta_i(t)| < \alpha + t\Delta(\Omega, D), \quad t \in [0, T^*). \tag{5.23}$$

• Step B ($T^* < \infty$): Suppose $T^* = \infty$. Then,

$$|\theta_i(t)| < \alpha + t\Delta(\Omega, D), \quad t \in [0, \infty), \quad i = 1, \cdots, N.$$

Letting $t \to \infty$, we see that the left-hand side is bounded below by zero, whereas the right-hand side becomes $-\infty$, which is a contradiction. Hence,

$$T^* < \infty \quad \text{and} \quad |\theta_i(t)| < \alpha + t\Delta(\Omega, D) \ i = 1, \cdots, N, \ t \in [0, T^*). \tag{5.24}$$

• Step C (Θ enters $(\mathcal{J}(\alpha^\infty))^N$ before T^*). More precisely, we claim:

$$\alpha + T^*\Delta(\Omega, D) \le \alpha^\infty. \tag{5.25}$$

If (5.25) is true, then

$$\max_{1 \le i \le N} |\theta_i(T^*)| \le \alpha + T^*\Delta(\Omega, D) \le \alpha^\infty,$$

i.e., the flow $\Theta_i(T^*)$ lies in the interval $(\mathcal{J}(\alpha^\infty))^N$.

Proof of claim (5.25)*:* Suppose to the contrary that (5.25) is not true, i.e.,

$$\alpha + T^*\Delta(\Omega, D) > \alpha^\infty.$$

Letting $t \to T^*-$ in (5.24), we have

$$|\theta_i(T^*)| \le \alpha + T^*\Delta(\Omega, D), \quad 1 \le i \le N.$$

⋄ Step C.1: For all $i \in \mathcal{N}$, assume that

$$|\theta_i(T^*)| < \alpha + T^*\Delta(\Omega, D). \tag{5.26}$$

By the continuity of Θ, there exists $\delta' > 0$ such that

$$|\theta_i(t)| < \alpha + t\Delta(\Omega, D), \quad i = 1, \cdots, N, \ t \in (0, T^* + \delta'),$$

i.e., $T^* + \delta' \in \mathcal{T}$, which contracts the fact that $T^* = \sup \mathcal{T} < \infty$.

⋄ Step C.2: From the result of Step C.1, (5.26) does not hold. Thus, there exists M such that

$$|\theta_M(T^*)| = \alpha + T^*\Delta(\Omega, D) \in (\alpha^\infty, \alpha), \quad |\theta_i(T^*)| \le |\theta_M(T^*)|, \quad \forall i. \tag{5.27}$$

Then relation (5.27) yields

$$|(SI)(\theta_M(T^*))| \ge |(SI)(\alpha^\infty)|. \tag{5.28}$$

We now use (5.28) to obtain

$$\begin{aligned}
\frac{d|\theta_M|}{dt}\Big|_{t=T^*} &\le |\omega_M| + K\sum_{j=1}^N c_{jM} S(|\theta_M(T^*)|) I(|\theta_j(T^*)|) \\
&\le |\omega_M| + K\sum_{j=1}^N c_{jM} S(|\theta_M(T^*)|) I(|\theta_M(T^*)|) \\
&= |\omega_M| - K\sum_{j=1}^N c_{jM} |S(|\theta_M(T^*)|) I(|\theta_M(T^*)|)| \\
&< |\omega_M| - K d_M |(SI)(\alpha^\infty)| \\
&\le \Delta(\Omega, D).
\end{aligned}$$

From the continuity of $\dot\theta_M$, there also exists $t^* < T^*$ such that

$$\frac{d|\theta_M|}{dt} \le \Delta(\Omega, D) \text{ for } t \in (t^*, T^*). \tag{5.29}$$

Integrating (5.29) from t^* to T^* yields

$$|\theta_M(t^*)| \ge |\theta_M(T^*)| + \Delta(\Omega, D)(t^* - T^*) = \alpha + t^*\Delta(\Omega, D).$$

However, this contradicts (5.23); hence,

$$\max_{1\le i\le N} |\theta_i(T^*)| \le \alpha + T^*\Delta(\Omega, D) \le \alpha^\infty.$$

Finally, t_e is chosen so that the following is satisfied:

$$\alpha + t_e\Delta(\Omega, D) = \alpha^\infty, \quad \text{i.e.,} \quad t_e := -\frac{\alpha - \alpha^\infty}{\Delta(\Omega, D)}.$$

□

5.2. **ℓ^1-contractivity of the Winfree flow.** In this subsection, we study the ℓ^1-contraction property of the Winfree flow. For all-to-all networks, similar results can be found in [21].

Proposition 5.2. *Suppose conditions* (2.6)-(2.7) *hold, and let* $\Theta = \Theta(t)$ *and* $\tilde{\Theta} = \tilde{\Theta}(t)$ *be two global smooth solutions to* (1.1)-(1.2) *with initial data* Θ_0 *and* $\widetilde{\Theta}_0$ *satisfying*

$$\alpha \in (0, \theta^*), \quad \Theta_0, \widetilde{\Theta}_0 \in (\mathcal{J}(\alpha))^N, \quad \text{and} \quad K > \frac{\|\Omega\|_\infty}{\min\limits_{1\le i\le N} d_i} \frac{1}{|(SI)(\alpha^\infty)|}.$$

Then there exists a $t_e \in [0, \infty)$ *and negative constants* λ_l *and* λ_u *such that*

$$\begin{aligned} e^{K\lambda_l(t-t_e)}\|\Theta(t_e) - \widetilde{\Theta}(t_e)\|_1 &\le \|\Theta(t) - \widetilde{\Theta}(t)\|_1 \\ &\le e^{K\lambda_u(t-t_e)}\|\Theta(t_e) - \widetilde{\Theta}(t_e)\|_1, \quad t \ge t_e, \end{aligned} \tag{5.30}$$

where t_e *is given in Proposition 5.1, and the negative constants* λ_l *and* λ_u *are defined by*

$$\begin{aligned} \lambda_l &:= (\max_{1\le i\le N} d_i)((S'I)(0) - (SI')(\alpha^\infty)) < 0, \\ \lambda_u &:= (\min_{1\le i\le N} d_i)(SI)'(\alpha^\infty) < 0. \end{aligned} \tag{5.31}$$

Proof. Note that if $\Theta(t) = \widetilde{\Theta}(t)$, then the desired inequalities hold. Thus, without loss of generality, we assume that

$$\Theta(t) \ne \widetilde{\Theta}(t), \quad t \ge t_e.$$

• Case A (Second inequality in (5.30)): The second inequality can be rewritten as follows:

$$\|\Theta(t) - \tilde{\Theta}(t)\|_1 e^{-K\lambda_u t} \le \|\Theta(t_e) - \tilde{\Theta}(t_e)\|_1 e^{-K\lambda_u t_e}, \quad t \ge t_e. \tag{5.32}$$

Set

$$\mathcal{L}_u(t) := \|\Theta(t) - \tilde{\Theta}(t)\|_1 e^{-K\lambda_u t}.$$

We claim that

$$\mathcal{L}_u \ \text{ is non-increasing on } [t_e, \infty). \tag{5.33}$$

Proof of claim (5.33): Since $\mathcal{L}_u$ is continuous, it suffices to show that for any $t_0 \in (t_e, \infty)$, there exists a $\delta > 0$ such that $\mathcal{L}(t_0) \ge \mathcal{L}(t)$ for all $t \in (t_0, t_0 + \delta)$.

Let $(s_1, \ldots, s_N) \in \{-1, 1\}^N$ be an N-tuple of signatures. Note that

$$\begin{aligned} &\sum_{i=1}^N s_i(\theta_i - \tilde{\theta}_i)(t_0) = \|\Theta(t_0) - \tilde{\Theta}(t_0)\|_1 \\ &\qquad \iff \quad s_i(\theta_i - \tilde{\theta}_i)(t_0) \ge 0 \\ &\qquad \iff \quad s_i = \begin{cases} 1 \text{ or } -1, & \text{if } (\theta_i - \tilde{\theta}_i)(t_0) = 0, \\ \operatorname{sgn}(\theta_i - \tilde{\theta}_i)(t_0), & \text{if } (\theta_i - \tilde{\theta}_i)(t_0) \ne 0. \end{cases} \end{aligned} \tag{5.34}$$

We use the mean-value theorem for two-variable functions to obtain

$$
\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^N s_i(\theta_i-\tilde\theta_i)\cdot e^{-K\lambda_u t}\right)\Big|_{t_0}\cdot e^{K\lambda_u t_0}\\
&= K\sum_{1\le i,j\le N} s_i c_{ji}\Big(S(\theta_i)I(\theta_j)-S(\tilde\theta_i)I(\tilde\theta_j)\Big)-K\lambda_u\sum_{i=1}^N s_i(\theta_i-\tilde\theta_i)\\
&= K\sum_{1\le i,j\le N} s_i c_{ji}\Big(S'(\theta_i^*)I(\theta_j^*)(\theta_i-\tilde\theta_i)+S(\theta_i^*)I'(\theta_j^*)(\theta_j-\tilde\theta_j)\Big)\\
&- K\lambda_u\sum_{i=1}^N s_i(\theta_i-\tilde\theta_i)\\
&= K\sum_{i=1}^N \underbrace{\Big\{\Big[\sum_{j=1}^N c_{ji}\Big(S'(\theta_i^*)I(\theta_j^*)+S(\theta_j^*)I'(\theta_i^*)s_is_j\Big)\Big]-\lambda_u\Big\}}_{=:\mathcal{I}_1} s_i(\theta_i-\tilde\theta_i)\Big|_{t=t_0}.
\end{aligned}
\tag{5.35}
$$

Here, $\theta_i^*,\theta_j^*\in(-\alpha^\infty,\alpha^\infty)$. The symmetry of c_{ij} and $\frac{1}{s_i}=s_i$ is used to show that

$$
\begin{aligned}
\sum_{i,j} s_ic_{ji}S(\theta_i^*)I'(\theta_j^*)(\theta_j-\tilde\theta_j) &= \sum_{i,j} s_jc_{ij}S(\theta_j^*)I'(\theta_i^*)(\theta_i-\tilde\theta_i)\\
&= \sum_{i,j} s_js_ic_{ji}S(\theta_j^*)I'(\theta_i^*)s_i(\theta_i-\tilde\theta_i).
\end{aligned}
$$

Note that the terms inside the summation of $\mathcal{I}_1$ can be estimated as follows.

- (Estimate of $S'(\theta_i^*)I(\theta_j^*)$): In this case, we use the monotonicity of S' and I:

$$
S'(0)\le S'(\theta_i^*)\le S'(\alpha^\infty)<0,\qquad I(0)\ge I(\theta_j^*)\ge I(\alpha^\infty)>0,
\tag{5.36}
$$

to obtain

$$
S'(\theta_i^*)I(\theta_j^*)\le S'(\alpha^\infty)I(\alpha^\infty)<0.
\tag{5.37}
$$

- (Estimate of $S(\theta_j^*)I'(\theta_i^*)$): Again, we use the monotonicity of S and I':

$$
S(\alpha^\infty)\le -|S(\theta_j^*)|\le S(0)\le 0,\qquad I'(\alpha^\infty)\le -|I'(\theta_i^*)|\le I'(0)\le 0,
$$

to obtain

$$
0<|S(\theta_j^*)I'(\theta_i^*)|\le S(\alpha^\infty)I'(\alpha^\infty).
\tag{5.38}
$$

For the sign of $\mathcal{I}$, (5.35), (5.37), and (5.38) are used to obtain

$$
\begin{aligned}
\mathcal{I}_1 &= \Big[\sum_{j=1}^N c_{ji}\Big(S'(\theta_i^*)I(\theta_j^*)+S(\theta_j^*)I'(\theta_i^*)s_is_j\Big)\Big]-(\min_{1\le i\le N} d_i)(SI)'(\alpha^\infty)\\
&\le \sum_{j=1}^N c_{ji}\Big[(S'I)(\alpha^\infty)+(SI')(\alpha^\infty))\Big]-(\min_{1\le i\le N} d_i)(SI)'(\alpha^\infty)\\
&= (d_i-\min_{1\le i\le N} d_i)(SI)'(\alpha^\infty)\le 0.
\end{aligned}
\tag{5.39}
$$

Next, s_i is chosen to satisfy

$$\sum_{i=1}^N s_i(\theta_i - \tilde{\theta}_i)(t_0) = \|\Theta(t_0) - \tilde{\Theta}(t_0)\|_1$$

as in (5.34). Then, in (5.35), we use (5.39) to obtain

$$\frac{d\mathcal{L}_u}{dt}\Big|_{t=t_0} = Ke^{-K\lambda_u t_0} \times \sum_{i=1}^N \mathcal{I}_1(t_0)|\theta_i(t_0) - \tilde{\theta}_i(t_0)| \le 0.$$

This proves (5.32).

• Case B (First inequality in (5.30)): Note that the first inequality can be rewritten as

$$\|\Theta(t_e) - \tilde{\Theta}(t_e)\|_1 e^{-K\lambda_l t_e} \le \|\Theta(t) - \tilde{\Theta}(t)\|_1 e^{-K\lambda_l t}, \quad t \ge t_e. \tag{5.40}$$

Set

$$\mathcal{L}_l(t) := \|\Theta(t) - \tilde{\Theta}(t)\|_1 e^{-K\lambda_l t}.$$

To prove (5.40), it suffices to show that the functional $\mathcal{L}_l$ is non-decreasing in the time-interval $[t_e, \infty)$. As in (5.35), we use (5.31) to obtain

$$\begin{aligned}
&\frac{d}{dt}\left(\sum_{i=1}^N s_i(\theta_i - \tilde{\theta}_i)\cdot e^{-K\lambda_l t}\right)\Bigg|_{t=t_0} \cdot e^{K\lambda_l t_0} \\
&= K\sum_{1\le i,j\le N} s_i c_{ji}\Big(S'(\theta_i^*)I(\theta_j^*)(\theta_i - \tilde{\theta}_i) + S(\theta_i^*)I'(\theta_j^*)(\theta_j - \tilde{\theta}_j)\Big) \\
&- K\lambda_l \sum_{i=1}^N s_i(\theta_i - \tilde{\theta}_i) \\
&= K\sum_{i=1}^N \underbrace{\left[\left(\sum_{j=1}^N c_{ji}S'(\theta_i^*)I(\theta_j^*)\right) - (\max_{1\le i\le N} d_i)S'(0)I(0)\right]}_{=:\mathcal{I}_2} s_i(\theta_i - \tilde{\theta}_i)\Big|_{t=t_0} \\
&+ K\sum_{i=1}^N \underbrace{\left[\left(\sum_{j=1}^N c_{ji}S(\theta_j^*)I'(\theta_i^*)s_i s_j\right) + (\max_{1\le i\le N} d_i)S(\alpha^\infty)I'(\alpha^\infty)\right]}_{=:\mathcal{I}_3} s_i(\theta_i - \tilde{\theta}_i)\Big|_{t=t_0},
\end{aligned} \tag{5.41}$$

for some $\theta_i^*, \theta_j^* \in \mathcal{J}(\alpha^\infty)$. Next, we estimate the signs of $\mathcal{I}_i,\ i = 2, 3$.

• (Estimate of $\mathcal{I}_2$): It follows from (5.36) that

$$0 > S'(\theta_i^*)I(\theta_j^*) \ge S'(0)I(0).$$

This relation is used to obtain

$$\begin{aligned}
\mathcal{I}_2 &= \left(\sum_{j=1}^N c_{ji}S'(\theta_i^*)I(\theta_j^*)\right) - (\max_{1\le i\le N} d_i)S'(0)I(0) \\
&\ge \Big(d_i - (\max_{1\le i\le N} d_i)\Big)S'(0)I(0) \ge 0.
\end{aligned} \tag{5.42}$$

• (Estimate of $\mathcal{I}_3$): It follows from (5.38) that

$$c_{ji}S(\theta_j^*)I'(\theta_i^*)s_is_j \geq -c_{ji}S(\alpha^\infty)I'(\alpha^\infty).$$

This yields

$$\begin{aligned}\mathcal{I}_3 &= \Big(\sum_{j=1}^N c_{ji}S(\theta_j^*)I'(\theta_i^*)s_is_j\Big) + (\max_{1\leq i\leq N} d_i)S(\alpha^\infty)I'(\alpha^\infty) \\ &\geq \Big((\max_{1\leq i\leq N} d_i) - d_i\Big)S(\alpha^\infty)I'(\alpha^\infty) \geq 0.\end{aligned} \tag{5.43}$$

It follows from (5.41), (5.42), and (5.43) that

$$\frac{d}{dt}\left(\sum_{i=1}^N s_i(\theta_i - \tilde{\theta}_i)\cdot e^{-K\lambda_l t}\right)\Bigg|_{t=t_0} = Ke^{-K\lambda_l t_0} \times \sum_{i=1}^N (\mathcal{I}_2 + \mathcal{I}_3)s_i(\theta_i - \tilde{\theta}_i)\Big|_{t=t_0} \geq 0.$$

Then, by the same logic employed in Case A, we have

$$\frac{d\mathcal{L}_l}{dt}\Big|_{t=t_0} = Ke^{-K\lambda_l t_0} \times \sum_{i=1}^N (\mathcal{I}_2(t_0) + \mathcal{I}_3(t_0))|\theta_i(t_0) - \tilde{\theta}_i(t_0)| \geq 0.$$

Hence, the first inequality holds. □

5.3. **Emergence of an equilibrium.** In this subsection, we present the unique existence of an equilibrium for (1.1)-(1.2) in the region $(\mathcal{J}(\alpha^\infty))^N$.

Theorem 5.1. (Emergence of COD) *Suppose conditions* (2.6) *and* (2.7) *hold, and let* $\Theta = \Theta(t)$ *be a global smooth solution to* (1.1)-(1.2) *satisfying*

$$\alpha \in (0, \theta^*), \quad \Theta_0 \in \mathcal{J}(\alpha)^N, \qquad \text{and} \quad K > \frac{\|\Omega\|_\infty}{\min\limits_{1\leq i\leq N} d_i}\frac{1}{|(SI)(\alpha^\infty)|}. \tag{5.44}$$

Then $\Theta(t)$ *converges to an equilibrium in the region* $\mathcal{J}(\alpha)^N$ *that is uniquely determined by only* Ω, K, *and* (c_{ij}), *but not by* Θ^0; *i.e., there exists a unique complete death state* $\phi := (\phi_1, \ldots, \phi_N) \in \mathcal{J}(\alpha)^N$ *such that*

$$\omega_i + K\sum_{j=1}^N c_{ji}S(\phi_i)I(\phi_j) = 0, \quad \lim_{t\to\infty}\theta_i(t) = \phi_i.$$

Moreover, the convergence of $\Theta(t)$ *to* Φ *is exponential with a decay rate in* $[K\lambda_l, K\lambda_u]$.

Proof. • Part A (Existence): Suppose (5.44) holds, and let $\Theta = \Theta(t)$ be a global solution to (1.1)-(1.2). For $T > 0$, we fix the shifted function Θ^T as follows:

$$\Theta^T(t) := \Theta(t + T), \quad t \geq 0.$$

Since system (1.1) is autonomous, it is easy to verify that Θ^T is a solution with initial data $\Theta(T)$. By Proposition 5.1, there exists a finite time $t_e \in [0, \infty)$ such that

$$\Theta(t), \Theta^T(t) \in \mathcal{J}(\alpha)^N, \quad t \geq t_e.$$

Applying Proposition 5.2 again, we see that

$$\|\Theta(t) - \Theta^T(t)\|_1 \leq e^{K\lambda_u(t-t_e)}\|\Theta(t_e) - \Theta(T + t_e)\|_1, \quad t \geq t_e. \tag{5.45}$$

In particular, if $T = 1$ and $t = n \geq t_e$,

$$\|\Theta(n) - \Theta(n+1)\|_1 \leq e^{K\lambda_u(n-t_e)}\|\Theta(t_e) - \Theta(1+t_e)\|_1.$$

This implies that the sequence $\{\Theta(n)\}$ is Cauchy in $(\mathbb{R}^d, \|\cdot\|_1)$; thus, it converges to some $\Phi \in \mathcal{J}(\alpha)^N$, i.e.,

$$\lim_{n\to\infty} \Theta(n) = \Phi.$$

Considering all $T \in [0,1)$, we see that $\Theta(t)$ itself converges to $\Phi = (\phi_1, \cdots, \phi_N)$:

$$\lim_{t\to\infty} \Theta(t) = \Phi. \tag{5.46}$$

On the other hand, it follows from Proposition 5.1 that Φ belongs to the closure of $\mathcal{R}(\alpha^\infty)$, and (5.46) implies

$$\begin{aligned} \omega_i + K\sum_{j=1}^N c_{ji}S(\phi_i)I(\phi_j) &= \lim_{t\to\infty}\left(\omega_i + K\sum_{j=1}^N c_{ji}S(\theta_i(t))I(\theta_j(t))\right) \\ &= \lim_{t\to\infty}\dot{\theta}_i(t). \end{aligned} \tag{5.47}$$

It follows from (5.45) with h instead of T that

$$\Big\|\frac{\Theta(t+h)-\Theta(t)}{h}\Big\|_1 \leq e^{K\lambda_u(t-t_e)}\Big\|\frac{\Theta(t_e+h)-\Theta(t_e)}{h}\Big\|_1, \quad t \geq t_e.$$

Letting $h \to 0$ yields

$$\|\dot{\Theta}(t)\|_l \leq e^{K\lambda_u(t-t_e)}\|\dot{\Theta}(t_e)\|_1, \quad t \geq t_e.$$

Again, we let $t \to \infty$ to obtain

$$\lim_{t\to\infty}\|\dot{\Theta}(t)\|_1 = \|\lim_{t\to\infty}\dot{\Theta}(t)\|_1 = 0, \quad \text{i.e.,} \quad \lim_{t\to\infty}\dot{\Theta}(t) = 0. \tag{5.48}$$

Finally, we combine (5.47) and (5.48) to show that the asymptotic limit Φ is, in fact, an equilibrium for (1.1)-(1.2):

$$\omega_i + K\sum_{j=1}^N c_{ji}S(\phi_i)I(\phi_j) = 0.$$

Thus, the asymptotic state emerging from the initial configuration in a large coupling regime is necessarily an equilibrium, i.e., the COD state. The boundary of $\mathcal{J}(\alpha)^N$ doesn't have any equilibrium points, so that Φ is in $\mathcal{J}(\alpha)^N$. Next, we show that the equilibrium in the region $\mathcal{R}(\alpha)$ is unique.

• Part B (Uniqueness): Let Φ and $\tilde{\Phi}$ be two equilibria states inside $\mathcal{J}(\alpha)^N$, and let $\Phi(t)$ and $\tilde{\Phi}$ be two solutions with initial data Φ and $\tilde{\Phi}$, respectively. Then,

$$\Phi(t) = \Phi, \quad \tilde{\Phi}(t) = \tilde{\Phi}, \quad t \geq 0.$$

Applying Proposition 5.2 to Φ and $\tilde{\Phi}$ yields

$$\|\Phi - \widetilde{\Phi}\|_1 \leq e^{K\lambda_u(t-t_e)}\|\Phi(t_e) - \widetilde{\Phi}(t_e)\|_1, \quad t \geq t_e. \tag{5.49}$$

If $\Phi \neq \widetilde{\Phi}$, then let $t \to \infty$ in (5.49) to find

$$\|\Phi - \widetilde{\Phi}\|_1 \leq 0, \quad \Phi = \widetilde{\Phi},$$

which is a contradiction. Hence, $\Phi = \widetilde{\Phi}$. □

6. Numerical simulations

In this section, we provide numerical simulations on the Winfree model with three symmetric networks (all-to-all, nearest neighbor, and star-shaped couplings). Numerical simulations on the all-to-all Winfree model are considered in [2, 34] for the phase diagram of each states. Our interest is to compare the numerical simulations with the analytic results which are given in previous sections. Throughout this section, we used the fourth order Runge-Kutta method with time step $\Delta t = 0.01$ and $N = 20$ for all numerical implementations. The sensitivity and influence functions were chosen as follows:

$$S(\theta) = -\sin\theta \quad \text{and} \quad I(\theta) = 1 + \cos\theta.$$

6.1. **Comparison of network structures.** In this subsection, we present numerical examples for symmetric network structures. We employ three kinds of interacting structures: an all-to-all network, nearest neighbor network, star-shaped network.

• All-to-all network: An all-to-all network is one in which all agents communicate mutually. Thus, the capacity matrix $\mathcal{C}_A$ for an all-to-all network is expressed by

$$\mathcal{C}_A = (c_{ji}), \quad c_{ji} = \frac{1}{N} \quad \text{for all} \quad i, j \in \mathcal{N}.$$

This yields that the degree of the network is given by $d_i = \sum_{j=1}^N c_{ji} = 1$ for all $i = 1, \cdots, N$.

• Nearest neighbor network: In a nearest neighbor network, each agent communicates with nearby agents, i.e., the ith agent interacts with both the $i-1$st and $i+1$st agents for $2 \le i \le N-1$, and the first and last agents communicate with each other. In this case, the capacity matrix $\mathcal{C}_R$ is given by

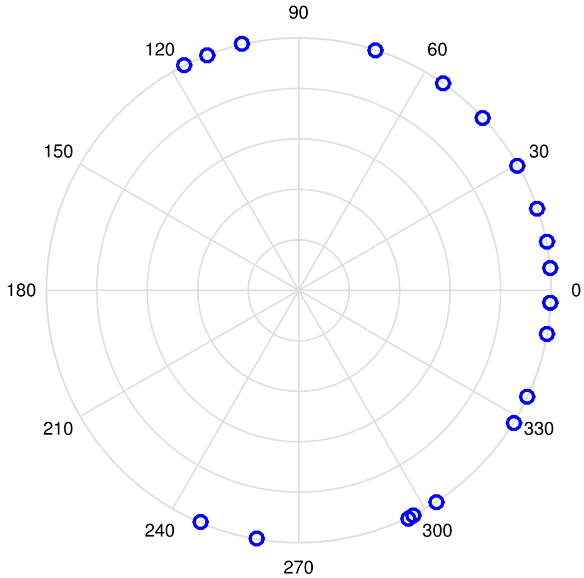
$$\mathcal{C}_N = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}.$$

In this network, the degree of network is determined by $d_i = \sum_{j=1}^N c_{ji} = 3$ for all $i = 1, \cdots, N$.

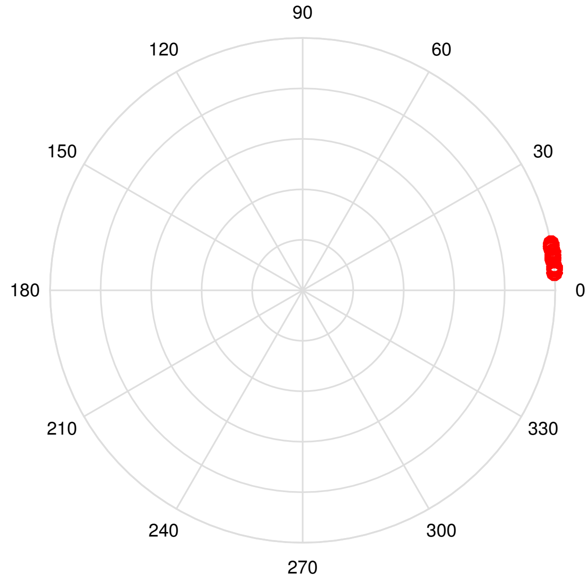
• Star-shaped network: In a star-shaped network, there exists a unique central agent; all other agents interact with only the central agent. Let the first agent play the role of the central agent. Then the capacity matrix $\mathcal{C}_S$ is given by

$$\mathcal{C}_S = \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 0 & 0 & \cdots & 1 & 0 \\ 1 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

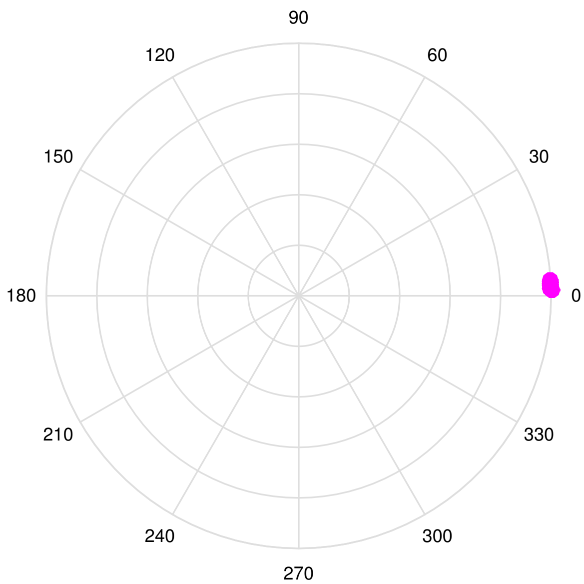
Note that when $N \ge 3$, the degree of the network $d_1 = \sum_{j=1}^N c_{j1} = N$ for the central agent is not equal to 2, whereas $d_i = \sum_{j=1}^N c_{ij} = 2$ for $i \ne 1$.



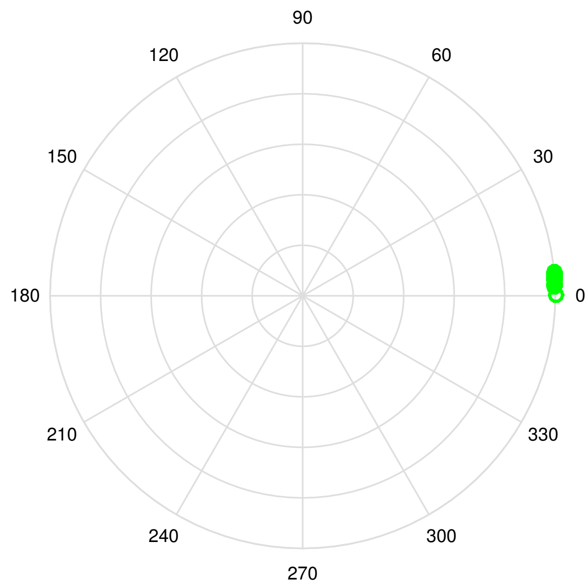
(a) Initial phase configuration Θ_0.



(b) $\lim_{t\to\infty}\Theta(t)$: all-to-all couplings.



(c) $\lim_{t\to\infty}\Theta(t)$: nearest neighbor couplings.



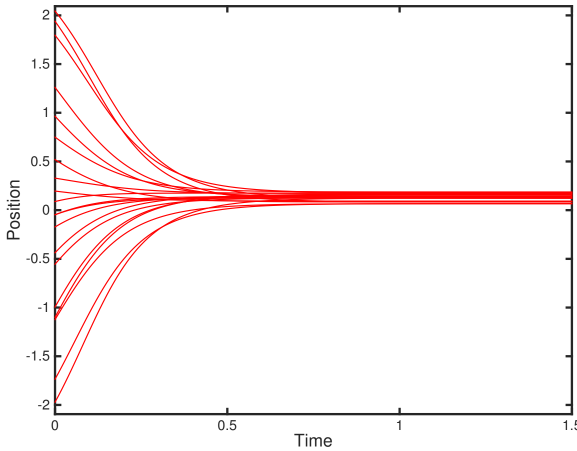
(d) $\lim_{t\to\infty}\Theta(t)$: star-shaped couplings.

FIGURE 3. Comparison of network structures.

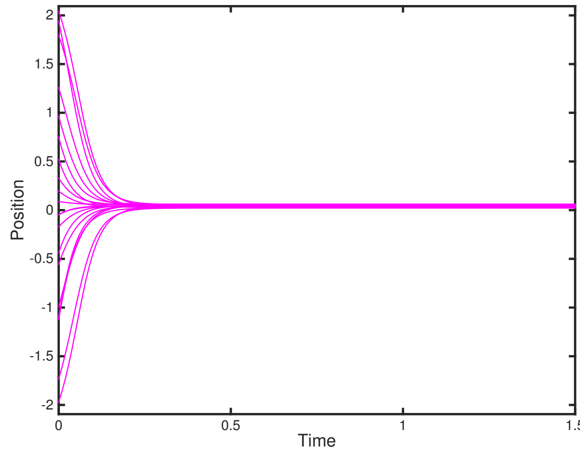
We set $\alpha = \frac{2\pi}{3}$ and randomly choose $N = 20$ oscillators so that the initial configuration satisfies

$$\Theta_0 = (\theta_{10}, \cdots, \theta_{N0}) \in (\mathcal{J}(\alpha))^N.$$

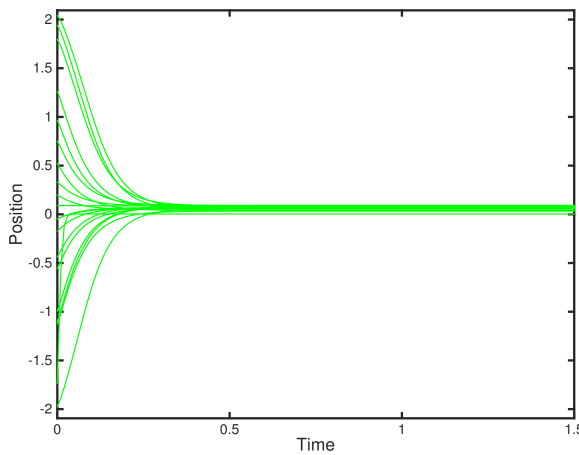
Since $(SI)(\alpha) = (SI)(\alpha^\infty) \approx -0.4330$, it follows that $\alpha^\infty \approx 0.2210$. Randomly choosing the natural frequency ω_i in $(\frac{1}{2}, \frac{3}{2})$ for all $i \in \mathcal{N}$ yields $\|\Omega\|_\infty = \frac{3}{2}$. The coupling strength



(a) Trajectory of oscillators with all-to-all networks.



(b) Trajectory of oscillators with nearest neighbor networks.



(c) Trajectory of oscillators with star-shaped networks.

FIGURE 4. Comparison of trajectories.

$K = 4$ is chosen to satisfy

$$K > \frac{\|\Omega\|_\infty}{\min\limits_{1\le i\le N} d_i}\frac{1}{|(SI)(\alpha^\infty)|} \approx 3.4641.$$

Figure 3 shows the initial configuration of the oscillators and the terminal configurations for each network structure. The trajectories of the solution for each network structure are shown in Figure 4.

6.2. **ℓ_1-stability.** In this subsection, we demonstrate the ℓ_1-contraction of the Winfree flow, which was discussed in Proposition 5.2. We use the previously determined settings and set $\alpha = \frac{2\pi}{3}$ and $K = 4$. Furthermore, the natural frequency ω_i is randomly chosen in the interval $(\frac{1}{2}, \frac{3}{2})$ for all $i \in \mathcal{N}$. We choose two distinct initial position Θ_0 and $\tilde{\Theta}_0$ in $(\mathcal{J}(\alpha))^N$ as shown in Figures 5(a) and 5(b). In Figure 6, we present the ℓ_1-contraction $\|\Theta - \tilde{\Theta}\|_1$

and log-plot $\log \|\Theta - \tilde{\Theta}\|_1$ for various network structures. Let Θ_A, Θ_N, and Θ_S denote the solutions for all-to-all, nearest neighbor, and star-shaped communications, respectively. The slopes in Figure 6(d) are given by

$$\begin{aligned}
&2(\log \|\Theta_A(1.5) - \tilde{\Theta}_A(1.5)\|_1 - \log \|\Theta_A(1) - \tilde{\Theta}_A(1)\|_1) \approx -7.9467\\
&2(\log \|\Theta_N(1.5) - \tilde{\Theta}_N(1.5)\|_1 - \log \|\Theta_N(1) - \tilde{\Theta}_N(1)\|_1) \approx -24.4947\\
&2(\log \|\Theta_S(1.5) - \tilde{\Theta}_S(1.5)\|_1 - \log \|\Theta_S(1) - \tilde{\Theta}_S(1)\|_1) \approx -16.1686.
\end{aligned}$$

In Proposition 5.2, the decay rate of the $\log \ell_1$-contraction is between $K\lambda_l$ and $K\lambda_u$. For all-to-all interacting network $\mathcal{C}_A$, the constants λ_l λ_u are give by

$$\begin{aligned}
\lambda_l &= (\max_{1\le i\le N} d_i)((S'I)(0) - (SI')(\alpha^\infty)) \approx -2.0480\\
\lambda_u &= (\min_{1\le i\le N} d_i)(SI)'(\alpha^\infty) \approx -1.8796,
\end{aligned}$$

where $\max\limits_{1\le i\le N} d_i = \min\limits_{1\le i\le N} d_i = 1$. Since $K = 4$, the decay rate for the all-to-all network satisfies

$$-8.1921 \approx K\lambda_l \le -7.9467 \le K\lambda_u \approx -7.5185.$$

For the nearest neighbor network $\mathcal{C}_N$, the maximum and minimum of degrees $\max\limits_{1\le i\le N} d_i = \min\limits_{1\le i\le N} d_i = 3$ for all $i = 1, \cdots, N$. This implies

$$\lambda_l \approx -6.1441, \quad \lambda_u \approx -5.6388.$$

so that

$$-24.5764 \approx K\lambda_l \le -24.4947 \le K\lambda_u \approx -22.5554.$$

On the other hand, for the star-shaped network $\mathcal{C}_S$, $d_1 = 20$ and $d_i = 2$ for all $i = 2, \cdots, N$, which yield

$$\lambda_l \approx -40.9607, \quad \lambda_u \approx -3.7592.$$

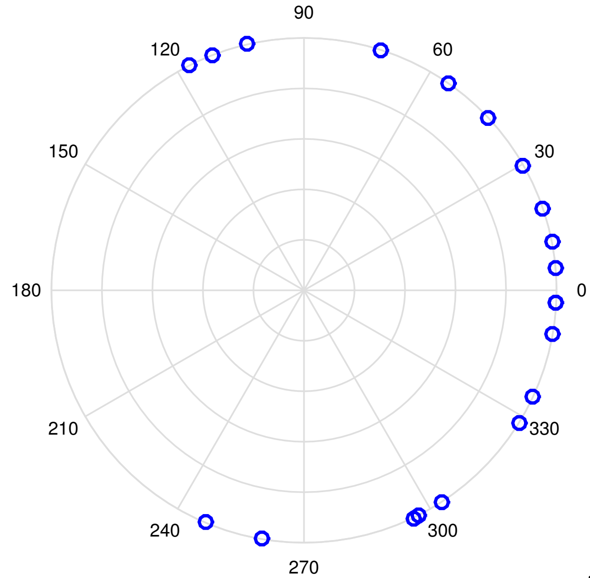
Hence, the decay rate is attained by

$$-163.8428 \approx K\lambda_l \le -16.1686 \le K\lambda_u \approx -15.0369.$$

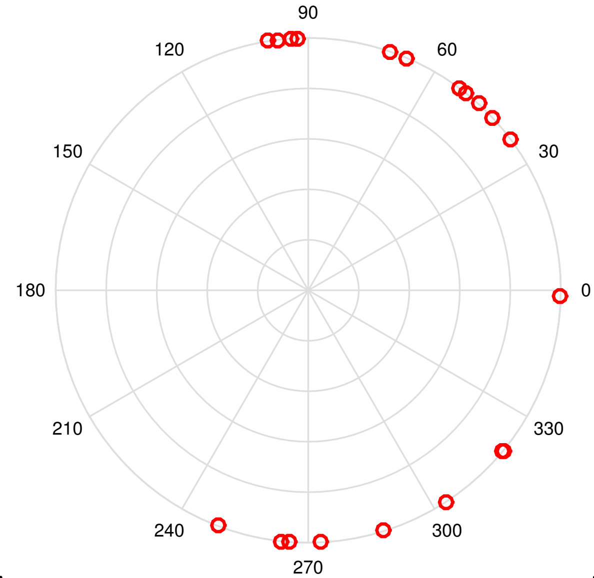
Therefore, for the three symmetric network structures, the numerical results agree with Proposition 5.2.

6.3. **Emergence of PPLS.** In this subsection, we show the emergence of PPLS by varying the coupling strength K. Initial configurations were chosen in $(-\pi, \pi)$. We arranged the natural frequencies in increasing order in $(0.8, 1.2)$. In Figure 7(a), the frequencies are equidistant. We plot the rotation numbers ρ_i for each oscillator $i \in \mathcal{N}$ by increasing the coupling strength $K \in (0.6, 0.8)$. In Figure 7, observe that there is no PPLS for the small coupling strength $K = 0.6$. PPLS emerges for the larger coupling strength $K = 0.7$. However, if the coupling strength increases from $K = 0.7$ to $K = 0.79$, the number of agents showing PPLS decreases. Finally for $K = 0.795$, the oscillators exhibit COD as shown in Figures 7 and 8. There also appear some monotonicity of the rotation numbers with respect to the natural frequencies, which can be explained as a following remark.

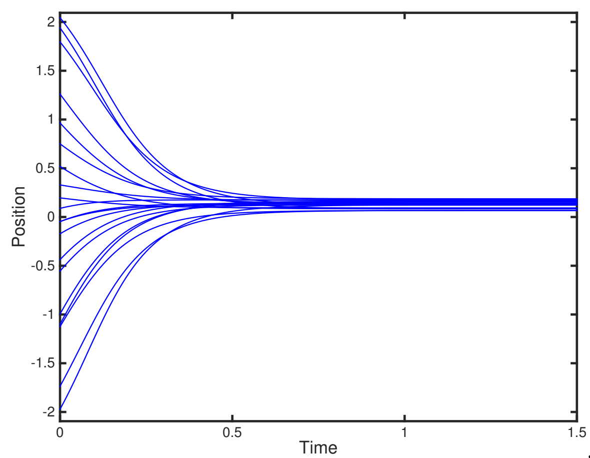
Remark 6.1. (Ordering principle of rotation numbers)
Suppose that the connection topology is all-to-all and $K \ge 0$. Let Θ be a solution to (1.1)



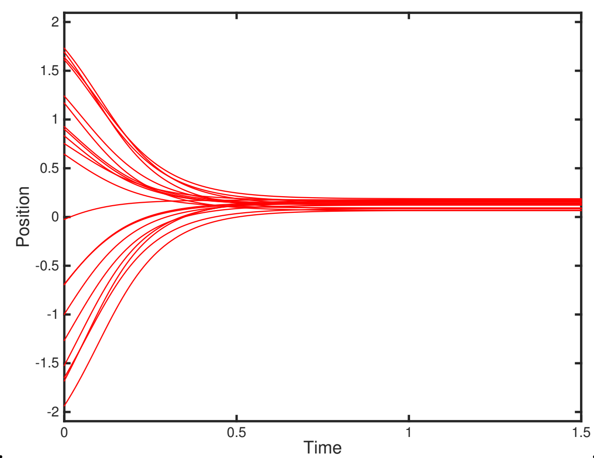
(a) Initial configuration of Θ_0



(b) Initial configuration of $\tilde{\Theta}_0$



(c) Trajectory of Θ



(d) Trajectory of $\tilde{\Theta}$

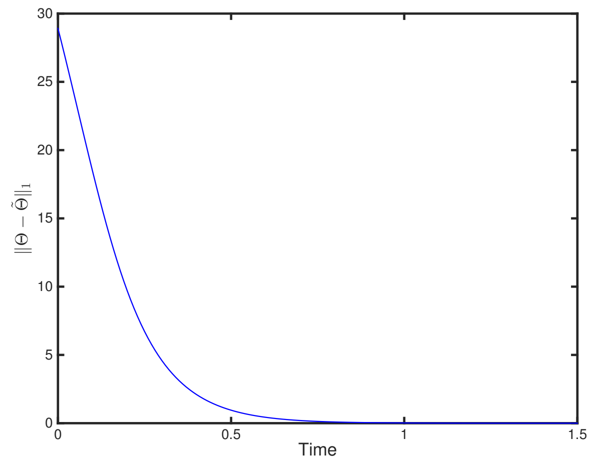
FIGURE 5. Initial configurations and trajectories of Θ and $\tilde{\Theta}$ for all-to-all communication networks.

with initial data Θ_0. Then, the rotation numbers are ordered according to the sizes of natural frequencies in the sense that

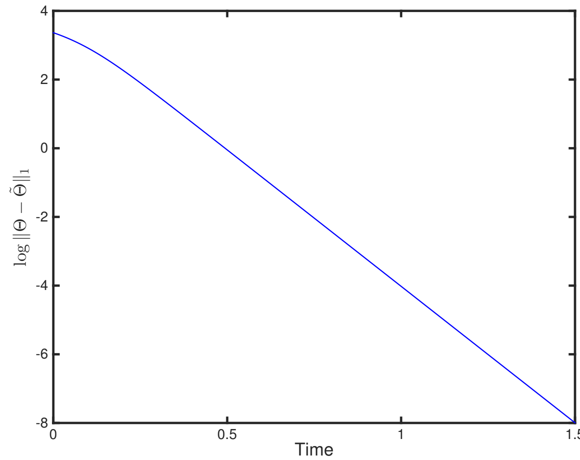
$$\omega_i \leq \omega_j \quad \textit{implies} \quad \rho_i \leq \rho_j, \ \textit{for any } i, j.$$

Proof. Assume that $\omega_i \leq \omega_j$, and rotation numbers ρ_i and ρ_j exist. Then, we can choose an integer m such that

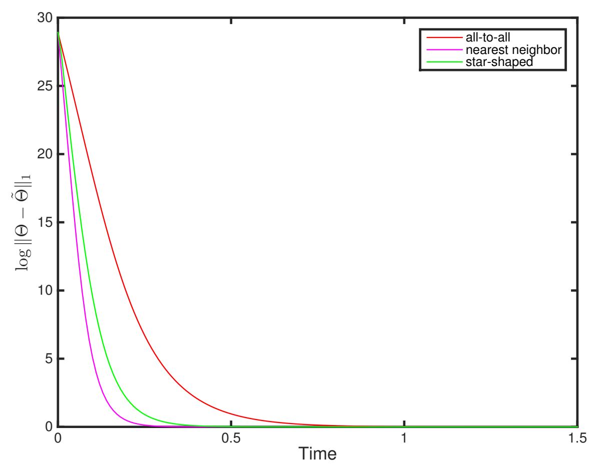
$$\theta_{j0} \geq \theta_{i0} + 2m\pi.$$



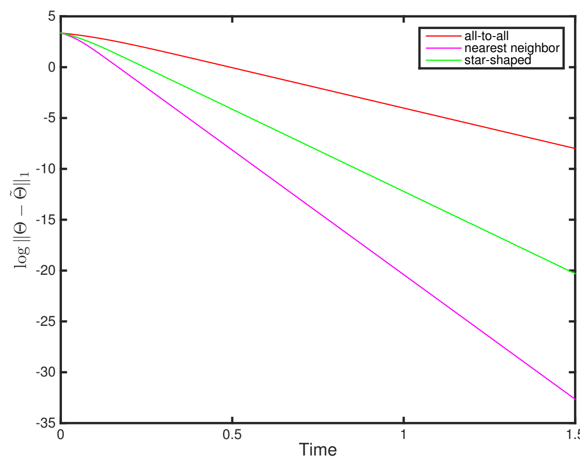
(a) ℓ_1-difference $\|\Theta - \tilde{\Theta}\|_1$ for the all-to-all network.



(b) $\log \ell_1$-difference $\log \|\Theta - \tilde{\Theta}\|_1$ for the all-to-all network.



(c) ℓ_1-difference $\|\Theta - \tilde{\Theta}\|_1$.



(d) $\log \ell_1$-difference $\log \|\Theta - \tilde{\Theta}\|_1$.

FIGURE 6. ℓ_1-contraction.

Suppose there is a time $t_* \geq 0$ satisfying $\theta_j(t_*) = \theta_i(t_*) + 2m\pi$. From the equation (1.1), we have

$$\frac{d}{dt}(\theta_j - \theta_i)(t_*) = \omega_j - \omega_i \geq 0.$$

This yields

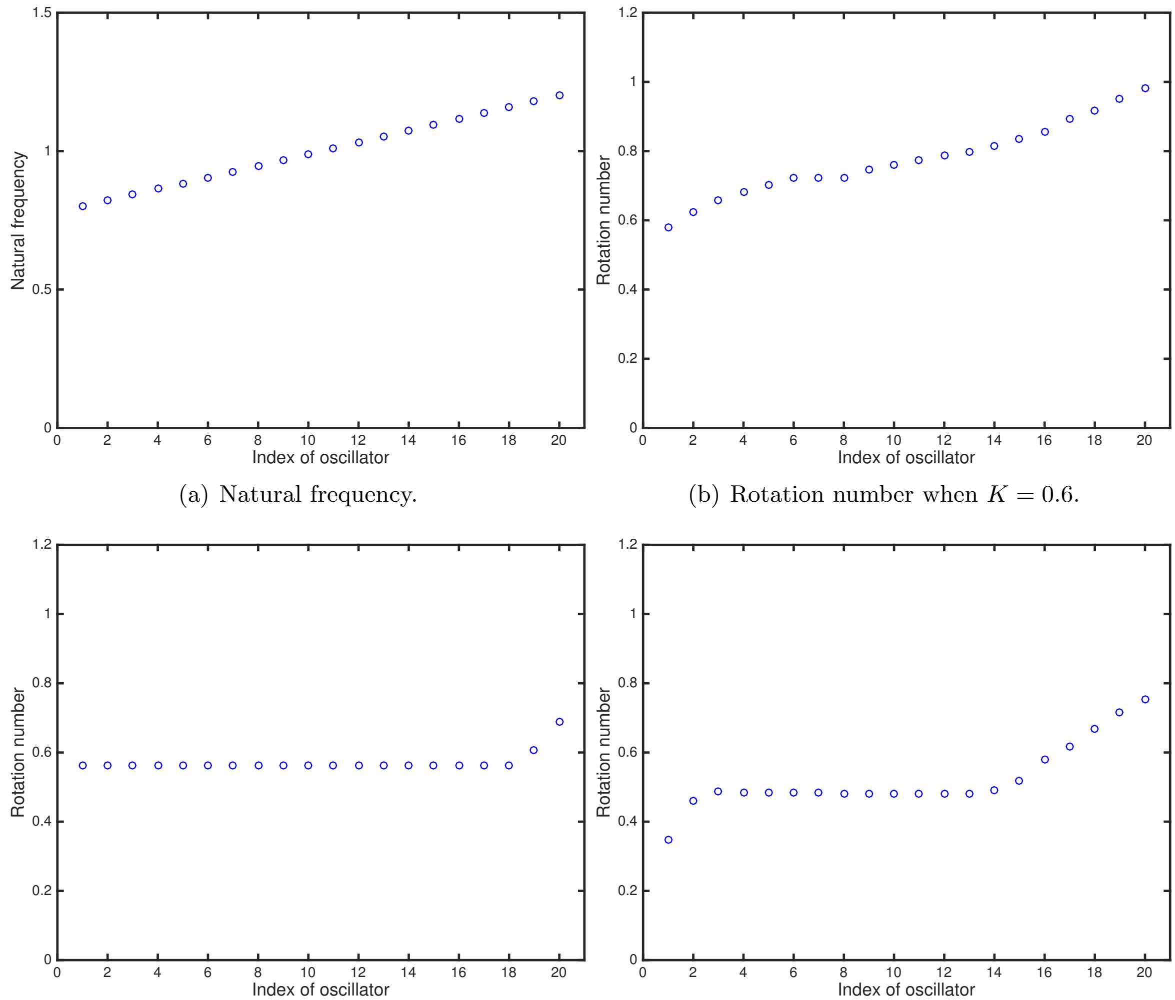
$$\theta_j(t) \geq \theta_i(t) + 2m\pi \quad \text{for all } t \geq 0.$$

Hence, we can get the following result.

$$\rho_j = \lim_{t\to\infty} \frac{\theta_j}{t} \geq \lim_{t\to\infty} \frac{\theta_i + 2m\pi}{t} = \rho_i.$$

□

We plot a schematic diagram of the rotation numbers of oscillators for varying coupling constant K in Figure 9. The same initial conditions are given for each simulations. As increasing of the coupling strength K, the rotation numbers for a part of oscillators coincide,



(a) Natural frequency.

(b) Rotation number when $K = 0.6$.

(c) Rotation number when $K = 0.7$

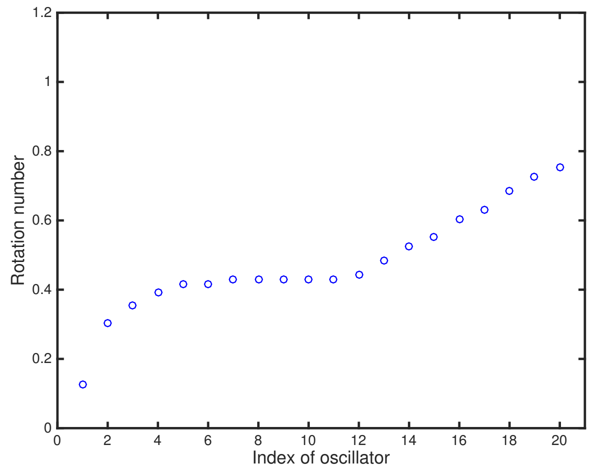
(d) Rotation number when $K = 0.75$

FIGURE 7. Rotation numbers with varying coupling strength K.

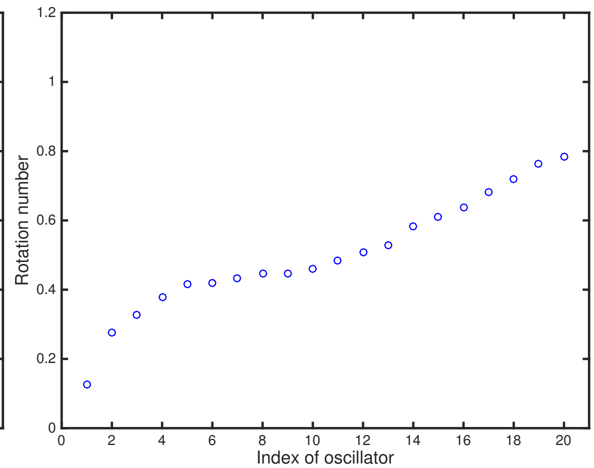
which signify the PPLS. If we keep raising the coupling strength, the rotation numbers show scattering, and finally, the phase states become COD with large coupling strength.

7. Conclusion

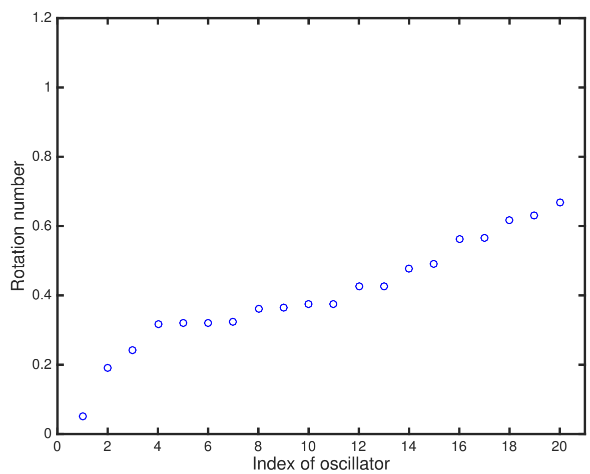
In this paper, we studied the emergent dynamics of Winfree oscillators on symmetric networks. For the emergence of complete or partial oscillator deaths (COD, POD), we provided sufficient frameworks in terms of connection topology, coupling strength, and coupling functions. Moreover, our results for COD and POD covered generic initial data in the case of special sensitivity and influence functions $S(\theta) = -\sin\theta$ and $I(\theta) = 1 + \cos\theta$, in large coupling regimes. We also provided three qualitative estimates; in particular, we considered the existence of an attractor with positive measures, exponential ℓ^1-contractivity, and the existence of an equilibrium inside the attractor of a large-coupling regime. The results in this paper generalized an earlier result for all-to-all couplings [21] .



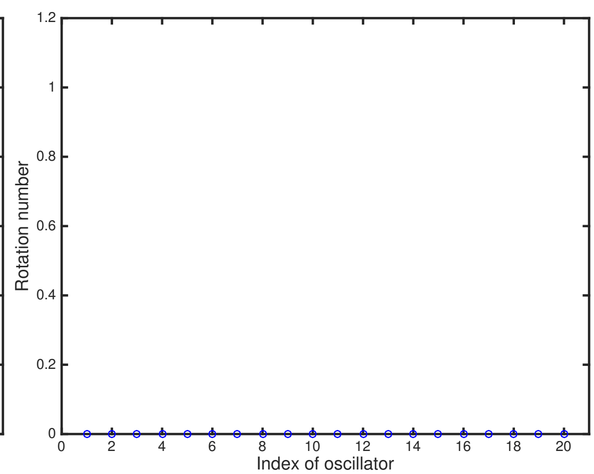
(a) Rotation number when $K = 0.78$.



(b) Rotation number when $K = 0.785$.



(c) Rotation number when $K = 0.79$.



(d) Rotation number when $K = 0.795$.

Figure 8. Rotation numbers with varying coupling strength K.

Acknowledgement

The work of S.-Y. Ha is supported by a NRF grant (2011-0015388) and the work of J. Park was supported by NRF(National Research Foundation of Korea) Grant funded by the Korean Government(NRF-2014-Fostering Core Leaders of the Future Basic Science Program)

References

References

[1] Acebron, J. A., Bonilla, L. L., Pérez Vicente, C. J. P., Ritort, F. and Spigler, R.: *The Kuramoto model: A simple paradigm for synchronization phenomena.* Rev. Mod. Phys. **77** (2005), 137–185.

[2] Ariaratnam, J. T. and Strogatz, S. H.: *Phase diagram for the Winfree model of coupled nonlinear oscillators.* Phys. Rev. Lett. **86** (2001), 4278–4281.

[3] Buck, J. and Buck, E.: *Biology of synchronous flashing of fireflies.* Nature **211** (1966), 562.

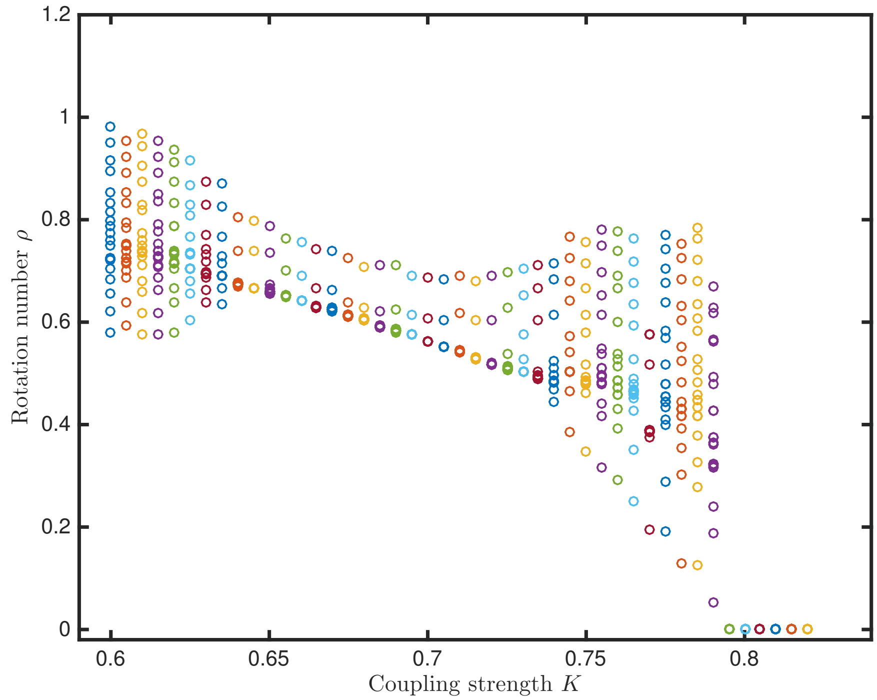


FIGURE 9. Rotation numbers with varying coupling strength K.

[4] Ballerini, M., Cabibbo, N., Candelier, R., Cavagna, A., Cisbani, E., Giardina, I., Lecomte, V., Orlandi, A., Parisi, G., Procaccini, A., Viale, M. and Zdravkovic, V.: *Interaction ruling animal collective behavior depends on topological rather than metric distance: Evidence from a field study.* Proc. Nat. Acad. Sci. **105** (2008), 1232–1237.

[5] Bellomo, N., Piccoli, B. and Tosin, A.: *Modeling crowd dynamics from a complex system viewpoint.* Math. Models Methods Appl. Sci. **22** (2012), 1230004–1230033.

[6] Bellomo, N., and Soler, J.: *On the mathematical theory of the dynamics of swarms viewed as complex systems.* Math. Models Methods Appl. Sci. **22** (2012), 1140006–1140035.

[7] Choi, Y.-P., Ha, S.-Y., Jung, S. and Kim, Y.: *Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model.* Physica D **241** (2012), 735–754.

[8] Carrillo, J. A., Klar, A., Martin, S. and Tiwari, S.: *Self-propelled interacting particle systems with roosting force.* Math. Mod. Meth. Appl. Sci. **20** (2010), 1533–1552.

[9] Chopra, N. and Spong, M. W.: *On exponential synchronization of Kuramoto oscillators.* IEEE Trans. Automatic Control **54** (2009), 353–357.

[10] Cucker, F. and Smale, S.: *Emergent behavior in flocks.* IEEE Trans. Automat. Control **52** (2007), 852–862.

[11] Degond, P. and Motsch, S.: *Large-scale dynamics of the Persistent Turing Walker model of fish behavior.* J. Stat. Phys. **131** (2008), 989–1022.

[12] Degond, P. and Motsch, S.: *Continuum limit of self-driven particles with orientation interaction.* Math. Mod. Meth. Appl. Sci. **18** (2008), 1193–1215.

[13] Dörfler, F. and Bullo, F.: *Synchronization in complex network of phase oscillators: A survey.* Automatica **50**, (2014), 1539–1564.

[14] Dörfler, F. and Bullo, F.: *On the critical coupling for Kuramoto oscillators.* SIAM. J. Appl. Dyn. Syst. **10** (2011), 1070–1099.

[15] Fornasier, M., Haskovec, J. and Toscani, G.: *Fluid dynamic description of flocking via Povzner-Boltzmann equation.* Physica D **240** (2011), 21–31.

[16] Fax, J. A. and Murray, R. M.: *Information flow and cooperative control of vehicle formations.* IEEE Trans. Automatic Control **49** (2004), 1465–1476.

[17] Ha, S.-Y., Kim, H. K., and Park, J.: *Remarks on the complete frequency synchronization of Kuramoto oscillators.* Submitted.

[18] Ha, S.-Y., Kim, H. K., and Ryoo, S. W.: *Emergence of phase-locked states for the Kuramoto model in a large coupling regime.* Submitted.

[19] Ha, S.-Y., Li, Z., and Xue, X.: *Formation of phase-locked states in a population of locally interacting Kuramoto oscillators.* J. Differential Equations. **255** (2013), 3053–3070.

[20] Ha, S.-Y. and Liu, J.-G.: *A simple proof of Cucker-Smale flocking dynamics and mean field limit.* Commun. Math. Sci. **7** (2009), 297–325.

[21] Ha, S.-Y., Park, J. and Ryoo, S. W.: *Emergence of phase-locked states for the Winfree model in a large coupling regime.* To appear in DCDS-A.

[22] Ha, S.-Y. and Tadmor, E.: *From particle to kinetic and hydrodynamic description of flocking.* Kinetic and Related Models **1** (2008), 415–435.

[23] Jadbabaie, A., Lin, J. and Morse, A.: *Coordination of groups of mobile autonomous agents using nearest neighbor rules.* IEEE Trans. Autom. Control **48** (2003), 988–1001.

[24] Katok, A. and Hasselblatt, B.: *Introduction to the modern theory of dynamical systems.* Cambridge university press, Cambridge, 1995.

[25] Kuramoto, Y.: *Chemical oscillations, waves and turbulence.* Springer-Verlag, Berlin. 1984.

[26] Kuramoto, Y.: *International symposium on mathematical problems in mathematical physics.* Lecture notes in theoretical physics. **30** (1975), 420.

[27] Leonard, N. E., Paley, D. A., Lekien, F., Sepulchre, R., Fratantoni, D. M. and Davis, R. E.: *Collective motion, sensor networks and ocean sampling.* Proc. IEEE **95** (2007), 48–74.

[28] Louca, S. and Atay, F. M.: *Spatially structured networks of pulse-coupled phase oscillators on metric spaces.*

[29] Motsch, S. and Tadmor, E.: *A new model for self-organized dynamics and its flocking behavior.* J. Stat. Phys. **144** (2011), 923–947.

[30] Nardulli, G., Mrinazzo, D., Pellicoro, M. and Stramaglia, S.: *Phase shifts between synchronized oscillators in the Winfree and Kuramoto models.* Available at "http://arxiv.org/pdf/physics/0402100.pdf".

[31] Paley, D. A., Leonard, N. E., Sepulchre, R., Grunbaum, D. and Parrish, J. K.: *Oscillator models and collective motion.* IEEE Control Systems. **27** (2007), 89–105.

[32] Park, J., Kim, H. and Ha, S.-Y.: *Cucker-Smale flocking with inter-particle bonding forces.* IEEE Tran. Automatic Control **55** (2010), 2617–2623.

[33] Perea, L., Gómez, G. and Elosegui, P.: *Extension of the Cucker-Smale control law to space flight formation.* J. Guidance, Control and Dynamics **32** (2009), 526–536.

[34] Quinn, D. D., Rand, R. H. and Strogatz, S.: *Singular unlocking transition in the Winfree model of coupled oscillators.* Physical Review E **75** (2007), 036218.

[35] Quinn, D. D., Rand, R. H. and Strogatz, S.: *Synchronization in the Winfree model of coupled nonlinear interactions.* A. ENOC 2005 Conference, Eindhoven, Netherlands, August 7–12, 2005 (CD-ROM).

[36] Ren. W. and Beard, R. W.: *Consensus seeking in multi-agent systems under dynamically changing interaction topologies.* IEEE Trans. Automatic Control **50** (2005), 655–661.

[37] Saber, R. O., Fax, J. A. and Murray, R. M.: *Consensus and cooperation in networked multi- agent systems.* Proc. of the IEEE **95** (2007), 215–233.

[38] Saber, R. O. and Murray, R. M.: *Consensus problems in networks of agents with switching topology and time-delays.* IEEE Trans. Automatic Control **49** (2004), 1520–1533.

[39] Topaz, C. M. and Bertozzi, A. L.: *Swarming patterns in a two-dimensional kinematic model for biological groups.* SIAM J. Appl. Math. **65** (2004), 152–174.

[40] Tanner, H. G., Jadbabaie, A. and Pappas, G. J.: *Flocking in fixed and switching networks.* IEEE Trans. Automatic Control **52** (2007), 863–868.

[41] Toner, J. and Tu, Y.: *Flocks, herds, and Schools: A quantitative theory of flocking.* Physical Review E. **58** (1998), 4828–4858.

[42] Vicsek, T., Czirók, A., Ben-Jacob, E., Cohen, I. and Schochet, O.: *Novel type of phase transition in a system of self-driven particles.* Phys. Rev. Lett. **75** (1995), 1226–1229.

[43] Winfree, A.: *24 hard problems about the mathematics of 24 hour rhythms.* Nonlinear oscillations in biology (Proc. Tenth Summer Sem. Appl. Math., Univ. Utah, Salt Lake City, Utah, 1978), pp. 93-126, Lectures in Appl. Math., 17, Amer. Math. Soc., Providence, R.I., 1979.

[44] Winfree, A.: *Biological rhythms and the behavior of populations of coupled oscillators* J. Theoret. Biol. **16** (1967), 15–42 .

(Seung-Yeal Ha)
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 151-747, Korea
E-mail address: syha@snu.ac.kr

(Dongnam Ko)
Department of Mathematical Sciences,
Seoul National University, Seoul 151-747, Korea
E-mail address: pyeong@snu.ac.kr

(Jinyeong Park)
Department of Mathematical Sciences,
Seoul National University, Seoul 151-747, Korea
E-mail address: pjy40@snu.ac.kr

(Sangwoo Ryoo)
Department of Mathematical Sciences,
Seoul National University, Seoul 151-747, Korea
E-mail address: sangwoo.ryoo.1@gmail.com