EMERGENCE OF PHASE-LOCKED STATES FOR THE WINFREE MODEL IN A LARGE COUPLING REGIME

SEUNG-YEAL HA, JINYEONG PARK, AND SANG WOO RYOO

Abstract. We study the large-time behavior of the globally coupled Winfree model in a large coupling regime. The Winfree model is the first mathematical model for the synchronization phenomenon in an ensemble of weakly coupled limit-cycle oscillators. For the dynamic formation of phase-locked states, we provide a sufficient framework in terms of geometric conditions on the coupling functions and coupling strength. We show that in the proposed framework, the emergent phase-locked state is the unique equilibrium state and it is asymptotically stable in an ℓ^1-norm; further, we investigate its configurational structure. We also provide several numerical simulations, and compare them with our analytical results.

1. Introduction

Collective coherent motions in complex systems, such as the aggregation of bacteria, flocking of birds, swarming of fish, herding of sheep, and flashing of fireflies, are often observed in biological systems [3, 4, 8, 10, 11, 12, 15, 20, 21, 26, 36, 38, 39, 40, 41]. Recently, research works on such collective coherent motions have received considerable attention from many scientific disciplines, such as applied mathematics, biology, computer science, statistical physics, and control theory, due to their diverse applications [5, 6, 16, 22, 25, 30, 29, 28, 33, 34, 35, 37] in relation with the decentralized control of multi-agents such as UAVs and robots. Following the pioneering works of Winfree [41] and Kuramoto [23, 24], several agent-based models were proposed and have been extensively studied analytically and numerically in the literature. Our main interest in this paper lies on the Winfree phase model, which has been proposed by Winfree [41] in 1967.

Consider an ensemble of N limit-cycle oscillators that can be visualized as point rotors on the unit circle $\mathbb{S}^1$, and let $x_i = e^{\sqrt{-1}\theta_i}$ be the spatial position of the i-th limit-cycle oscillator on the unit circle. In this situation, the Winfree phase model is the coupled first-order system of ODEs:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} S(\theta_i)I(\theta_j), \quad i = 1, \ldots, N, \tag{1.1}$$

where K is coupling strength, and Ω_i is the natural frequency of the i-th limit-cycle oscillator, which is a random variable drawn from some given distribution function $g = g(\Omega)$. Since we are only interested in an ensemble of a finite number of oscillators, the explicit

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form of g will not be employed in the analysis, throughout the paper. Detailed discussions on the coupling mechanism, and geometric conditions on the sensitivity function S and influence function I, will be given in Section 2.1.

The Winfree model (1.1) with the special pair $(S(\theta), I(\theta)) = (-\sin\theta, 1+\cos\theta)$ has been addressed by researchers, in [2, 27, 31, 32]. Compared to the extensive research [1, 7, 9, 13, 14, 17, 18, 19] on the Kuramoto model, the Winfree model (1.1) has rarely been treated in the synchronization community. This might be due to the lack of symmetries in (1.1), for example, the Winfree model does not exhibit a translation invariance which results in the lack of conservation of the total phase. Moreover, the aforementioned literature on (1.1) mostly deals with infinite ensembles of Winfree oscillators, via the integro-differential equation in the mean-field setting. Thus, as far as the authors know, there are no systematic studies on the synchronization of the finite-dimensional Winfree model itself. In particular, there is no analytical result on the existence, formation, and stability of phase-locked states (see [7, 9, 14] for the corresponding issues in the Kuramoto model). Like for the Kuramoto model [23, 24] in the mean-field setting ($N \to \infty$), we might expect similar dynamical features for the Winfree model as well. More precisely, as we increase the coupling strength K from zero to some large value continuously, the Winfree model in the mean-field limit might exhibit two phase-transition-like phenomena: First, when the coupling strength is smaller than some positive critical coupling strength K_c, the phase configuration will lie in *the incoherent state*. Thus, in this stage, there will be no coherent ordered motions in the configuration. However, as the coupling strength exceeds K_c, locally ordered motions in the configuration will emerge asymptotically (*emergence of partially phase-locked states*), in which the synchronizing oscillators and drifting oscillators coexist. Subsequently, as we further increase the coupling strength to the value of another critical coupling strength $K_{sc}(> K_c)$, all of the phase velocities (frequencies) of the oscillator will tend to the same value asymptotically (*emergence of phase-locked state*), so that the position configuration on the unit circle will rotate the unit circle like a train. In summary, for a given configuration, in the successive two limiting procedures $t \to \infty$ and $K \to \infty$, the configuration might exhibit the following phase-transitions:

$$\begin{aligned} \text{incoherent state} \quad &\Longrightarrow \quad \text{partially phase locked-state} \\ &\Longrightarrow \quad \text{phase-locked state} \quad \text{as } K \to \infty. \end{aligned}$$

Most physics literature focuses mainly on the overall dynamics of the ensemble near the first phase transition regime, around $K = K_c$. In contrast, engineering literature deals with the second phase transition, near $K = K_{sc}$. As far as the authors know, the explicit calculations of critical coupling strengths K_c and K_{sc} for the Winfree model (1.1) have not yet been performed.

In this paper, we are interested in the second phase transition, "*emergence of complete synchronization*" (see Definition 2.1). More precisely, we shall address the following question:

> "Under what conditions on K, S, I, and the initial data, can we expect emergence of a phase-locked state asymptotically?"

The main results of this paper are three-fold: First, we present a sufficient framework guaranteeing the emergence of phase-locked states from an initial configuration along the dynamic Winfree flow. Second, we show that two Winfree flows issued from different initial configurations are exponentially approaching one another in an ℓ^1-norm within the proposed framework. Third, we provide quantitative estimates on the finiteness of the collisions and the structure of the phase-locked state in the dynamic process.

The rest of this paper is divided into five sections. In Section 2, we briefly discuss the coupling mechanism in the Winfree model and its relation with the Kuramoto model. We also present a framework and main result for the complete synchronization to the Winfree model. In Section 3, we present several estimates on the existence of a positively invariant set and exponential contraction of the Winfree flow. In Section 4, we show the existence of a unique equilibrium state that is emergent from generic initial configurations, and provide its structural configuration. In Section 5, we present several numerical simulations and compare them with analytical results given in previous sections. Finally, Section 6 is devoted to the summary of our main results.

Notation. We set $\mathcal{N} := \{1, \cdots, N\}$, and for $\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N$ and $(\Omega_1, \cdots, \Omega_N) \in \mathbb{R}^N$, we set

$$||\Theta||_1 := \sum_{i=1}^{N} |\theta_i|, \qquad \Omega^\infty := \max_{1\le i\le N} |\Omega_i|.$$

2. Framework and main result

In this section, we briefly discuss Winfree's spirit in the modeling of weak couplings among limit-cycle oscillators in (1.1), and its relation with the Kuramoto phase model, which is the most frequently studied mathematical model for synchronization. We also provide a framework and main result for the complete synchronization.

We first recall two basic definitions to be used throughout the paper.

Definition 2.1. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a configuration whose dynamics is governed by (1.1).*

(1) The configuration $\Theta = \Theta(t)$ tends to the (strong) phase-locked state asymptotically if and only if its transversal phase differences tend to constant values: for $i \neq j \in \mathcal{N}$, there exists a constant value θ_{ij}^∞, such that

$$\lim_{t\to\infty} |\theta_i(t) - \theta_j(t)| = \theta_{ij}^\infty.$$

(2) The configuration $\Theta = \Theta(t)$ exhibits a complete (frequency) synchronization if and only if its transversal frequency differences tend to zero: for $i \neq j \in \mathcal{N}$, we have

$$\lim_{t\to\infty} |\dot\theta_i(t) - \dot\theta_j(t)| = 0.$$

Remark 2.1. *Note that in the definition of complete synchronization, we do not require any explicit decay condition for transversal frequency differences. However, we can show that for the Winfree model, the transversal frequencies tend to zero exponentially fast (see Proposition 3.2) in a large coupling regime. For the Winfree model, the emergence of complete synchronization is equivalent to the formation of a phase-locked state. Furthermore, we will show that the emergent phase-locked state is the unique equilibrium (see Theorem 2.1).*

2.1. The Winfree phase model.

In 1967, Arthur Winfree [41] proposed a coupled phase model for many weakly coupled limit-cycle oscillators. Limit-cycle oscillators can be visualized as point rotors, moving on the unit circle $\mathbb{R}^1$. In the absence of mutual interactions between oscillators, the phase velocity (frequency) is given by the natural frequency of the oscillator. Let $\theta_i = \theta_i(t)$ and Ω_i be the phase and natural frequency, respectively, of the i-th oscillator. Then, in the absence of mutual interactions, the dynamics of θ_i is governed by the following decoupled system of ODEs:

$$\dot{\theta}_i = \Omega_i, \quad \text{i.e.,} \quad \theta_i(t) = \theta_{i0} + \Omega_i t, \qquad i = 1, \cdots, N. \tag{2.2}$$

On the other hand, in the presence of mutual interactions, the dynamics (2.2) should be supplemented by adding frequency perturbation terms in the R.H.S. of (2.2), due to the weak interactions:

$$\dot{\theta}_i = \Omega_i + \hat{\omega}_i, \quad i = 1, \cdots, N. \tag{2.3}$$

Thus, the intriguing modeling question is that of how to model such frequency perturbations $\hat{\omega}_i$ in (2.3). For this, we next briefly discuss how Winfree is able to model frequency perturbations.

Consider an ensemble of many weakly interacting limit-cycle oscillators, so that the amplitude states of the oscillators are fixed, but the phases are dynamic variables. In [41], Winfree introduced two functions, measuring the weak phase interactions via the influence function $I = I(\theta)$ and the sensitivity (or response) function $S = S(\theta)$, and employed two simplifying assumptions for frequency perturbations:

- The stimulus I_c impinging on the i-th oscillator is the average of the weak influences contributed by all phases of the oscillators in the ensemble (mean-field interaction):
$$I_c(\Theta) := \frac{1}{N}\sum_{j=1}^{N} I(\theta_j).$$
In this case, two tacit assumptions are made. First, the influences of an individual are assumed to be propagated without attenuation, in a time much shorter than the average period of the oscillators. Second, they are additive in their effect.
- The frequency perturbation $\hat{\omega}_i$ of the i-th oscillator is proportional to the product of the sensitivity $S(\theta_i)$ and the average stimulus $I_c(\Theta)$:
$$\hat{\omega}_i = KS(\theta_i)I_c(\Theta) = \frac{K}{N}\sum_{j=1}^{N} S(\theta_i)I(\theta_j).$$

Based on the above two assumptions, Winfree proposed the following coupled ODE system:

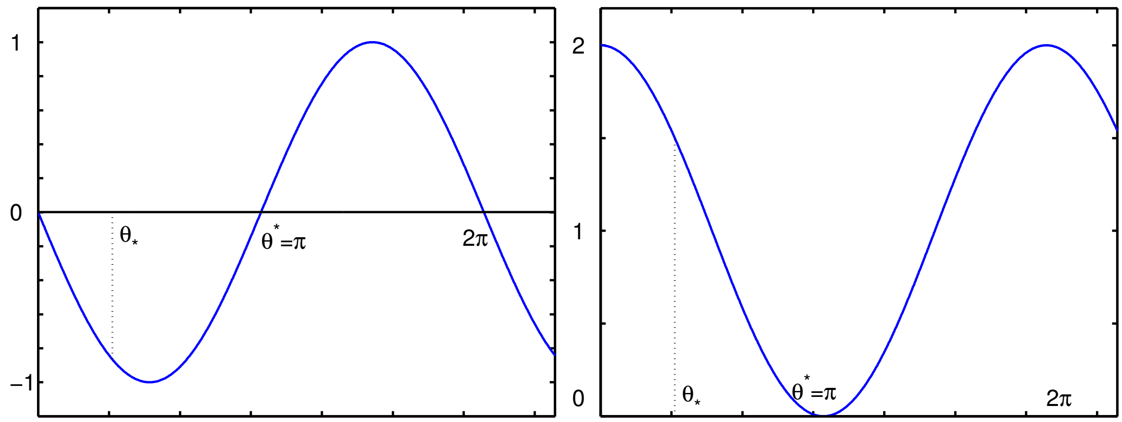
$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} S(\theta_i)I(\theta_j), \quad i = 1, \ldots, N.$$

2.2. **Relation with the Kuramoto model.** As noted in [40], the Kuramoto phase model can be cast into a generalized Winfree model. Recall the Kuramoto phase model [23, 24]:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \cdots, N. \tag{2.4}$$

By expanding the sinusoidal coupling term in the R.H.S. of (2.4), we can see that (2.4) can be rewritten as

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} \cos\theta_i \sin\theta_j + \frac{K}{N}\sum_{j=1}^{N} \sin\theta_i \times (-\cos\theta_j). \tag{2.5}$$



(a) $S(\theta) = -\sin\theta$ (b) $I(\theta) = 1 + \cos\theta$

FIGURE 1. Schematic Diagrams for $S(\theta)$ and $I(\theta)$

Then, by setting

$$(S_1(\theta), I_1(\theta)) := (\cos\theta, \sin\theta), \qquad (S_2(\theta), I_2(\theta)) := (\sin\theta, -\cos\theta),$$

system (2.5) can be rewritten as a generalized Winfree model:

$$\dot{\theta}_i = \Omega_i + \frac{K}{N}\sum_{j=1}^{N} S_1(\theta_i)I_1(\theta_j) + \frac{K}{N}\sum_{j=1}^{N} S_2(\theta_i)I_2(\theta_j).$$

Note that the pairs of sensitivity and influence functions are orthogonal, in the sense that

$$\langle (S_1(\theta), I_1(\theta)), (S_2(\theta), I_2(\theta)) \rangle = 0,$$

where $\langle \cdot \rangle$ is the standard ℓ^2-inner product in $\mathbb{R}^2$.

2.3. **Framework and main result.** In this subsection, we present a framework, employed in the complete synchronization for the Winfree model, and provide our main result.

Let S and I be the sensitivity and influence functions, respectively, employed in the coupling function of (1.1), satisfying the following structural conditions:

- $(\mathcal{A}1)$: The sensitivity function S is a 2π-periodic, $\mathcal{C}^2$, and odd function, and the influence function I is a 2π-periodic, $\mathcal{C}^2$, and even function:

$$\begin{aligned} &S(\theta + 2\pi) = S(\theta), \quad S(-\theta) = -S(\theta), \quad \theta \in \mathbb{R}, \\ &I(\theta + 2\pi) = I(\theta), \quad I(-\theta) = I(\theta), \quad \theta \in \mathbb{R}. \end{aligned} \tag{2.6}$$

- $(\mathcal{A}2)$: The sensitivity and influence functions satisfy some geometric conditions: there exist positive constants θ_* and θ^*, satisfying

$$0 < \theta_* < \theta^* < 2\pi,$$

such that,

$$\begin{aligned} &S \le 0 \quad \text{on } [0, \theta^*] \quad \text{and} \quad S' \le 0, \quad S'' \ge 0 \quad \text{on } [0, \theta_*], \\ &I \ge 0, \quad I' \le 0 \quad \text{on } [0, \theta^*], \quad \text{and} \quad I'' \le 0 \quad \text{on } [0, \theta_*], \\ &(SI)' < 0 \quad \text{on } (0, \theta_*), \quad (SI)' > 0 \quad \text{on } (\theta_*, \theta^*), \end{aligned} \tag{2.7}$$

where S' denotes the θ-derivative of S (see Figure 1 for schematic graph of S and I).

Note that $S(0) = 0$, and the following special pair (S, I), employed in [2, 27, 31, 32], satisfy the structural conditions in (2.6) and (2.7):

$$S(\theta) = -\sin\theta, \qquad I(\theta) = 1 + \cos\theta, \qquad \theta_* = \frac{\pi}{3}, \qquad \theta^* = \pi$$

Before we describe our main result, we introduce some notation. For a given $\alpha \in (0, \theta^*)$, consider the following equation on the interval $[0, \theta_*]$:

$$(SI)(x) = (SI)(\alpha), \quad x \in [0, \theta_*]. \tag{2.8}$$

Note that the conditions (2.6) and (2.7) yield the following geometric shape of the coupling function SI (see Figure 2):

$$\begin{aligned} &(SI)(0) = 0, \quad \theta_* = \mathrm{argmin}_{0\le\theta\le\theta^*}(SI)(\theta), \\ &(SI)(\theta) < 0 \quad \text{on } \theta \in (0, \theta^*), \quad (SI)(\theta^*) \le 0. \end{aligned} \tag{2.9}$$

Thus, the equation (2.8) has a unique solution α^∞ guaranteed by the relation (2.9). Moreover, for $\alpha \in (0, \theta_*], \alpha^\infty = \alpha$. For such α^∞, we define the coupling strength $K_e(\alpha^\infty)$ and a set $\mathcal{R}(\alpha)$ as follows:

$$\begin{aligned} &K_e(\alpha^\infty) := -\frac{\Omega^\infty}{S(\alpha^\infty)I(\alpha^\infty)} \quad \text{and} \\ &\mathcal{R}(\alpha) := \{\Theta = (\theta_1, \cdots, \theta_N) \in \mathbb{R}^N \mid \theta_i \in (-\alpha, \alpha), \ i = 1, \dots, N\}. \end{aligned} \tag{2.10}$$

Theorem 2.1. *Suppose that the conditions* (2.6) *and* (2.7) *hold, and for* $\alpha \in (0, \theta^*)$*, let* $\Theta = \Theta(t)$ *be a global smooth solution for the system* (1.1)*, satisfying*

$$\Theta_0 \in \overline{\mathcal{R}(\alpha)} \quad \textit{and} \quad K > K_e(\alpha^\infty).$$

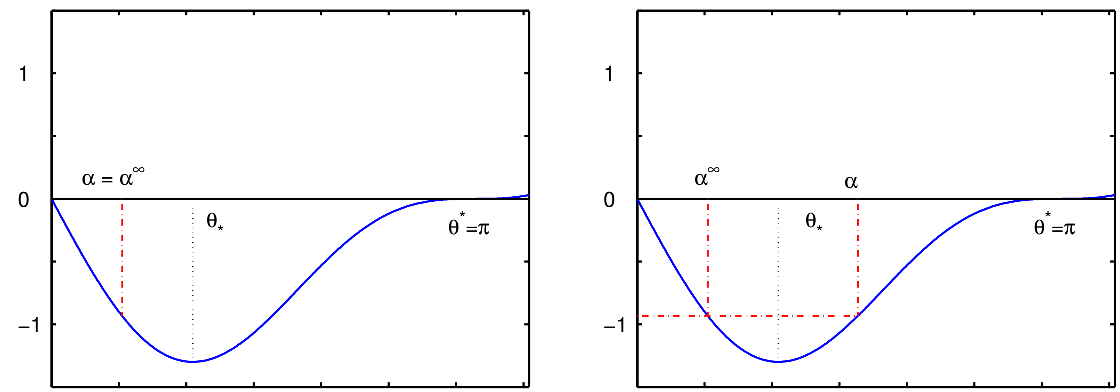
Then, $\Theta(t)$ *converges to a unique equilibrium state* $\Phi = (\phi_1, \cdots, \phi_N)$ *in the region* $\mathcal{R}(\alpha^\infty)$*, i.e., there exists a unique phase-locked state* $\Phi := (\phi_1, \dots, \phi_N) \in \mathcal{R}(\alpha^\infty)$*, such that*

$$\Omega_i + \frac{K}{N} S(\phi_i) \sum_{j=1}^N I(\phi_j) = 0, \qquad \lim_{t\to\infty} \Theta(t) = \Phi.$$

Proof. The proof will consist of three parts. First, we show that $\Theta(t)$ enters $\mathcal{R}(\alpha^\infty)$ within some finite time (see Proposition 3.1). Then, using the stability estimate in ℓ^1-distance (see Proposition 3.2), we show that any two solutions to (1.1) with configurations in $\mathcal{R}(\alpha^\infty)$ must converge to each other asymptotically. Finally, we show the existence of an equilibrium in $\mathcal{R}(\alpha^\infty)$ (see Section 4.1). Subsequently, the stability estimate implies that $\Theta(t)$ must converge to that phase-locked state, for the equilibrium itself is a solution to (1.1). □

3. Key estimates of the Winfree flow

In this section, we provide a priori key estimates on the positive invariance of $\mathcal{R}(\alpha^\infty)$, and stability estimates of the Winfree flow.



(a) Position of α^∞ for $\alpha \in (0, \theta_*)$ (b) Position of α^∞ for $\alpha \in (\theta_*, \theta^*)$

FIGURE 2. Determination of α^∞

3.1. **Positive invariance of $\mathcal{R}(\alpha^\infty)$.** In this subsection, we show that the set $\mathcal{R}(\alpha^\infty)$ defined in (2.10) is a positively invariant set, and that it also attracts neighboring configurations.

Lemma 3.1. (Positive invariance) *Suppose that the conditions* (2.6) *and* (2.7) *hold, and α and the coupling strength satisfy*

$$\alpha \in (0, \theta^*) \quad \textit{and} \quad K > K_e(\alpha^\infty).$$

Then, the set $\mathcal{R}(\alpha^\infty)$ is positively invariant along the Winfree flow (1.1)*:*

$$\Theta_0 \in \mathcal{R}(\alpha^\infty) \qquad \Longrightarrow \qquad \Theta(t) \in \mathcal{R}(\alpha^\infty), \quad t \in (0, \infty).$$

Proof. For a given $t \in (0, \infty)$, we introduce a maximal index $M = M(t)$, such that

$$M(t) := \mathrm{argmax}_{1 \le i \le N} |\theta_i(t)|.$$

We set

$$\mathcal{T} := \{T \in (0, \infty] : \ \Theta(t) \in \mathcal{R}(\alpha^\infty), \ t \in [0, T)\} \qquad \text{and} \qquad T^\infty := \sup \mathcal{T}.$$

Then, by the continuity of $\Theta(t)$ and the assumption that $\Theta_0 \in \mathcal{R}(\alpha^\infty)$, there exists a positive constant $\delta_1 > 0$ such that

$$\Theta(t) \in \mathcal{R}(\alpha^\infty) \quad \text{for } t \in [0, \delta_1).$$

Thus, the set $\mathcal{T}$ is nonempty, and T^∞ exists. We now claim that

$$T^\infty = \infty. \tag{3.11}$$

Once we prove (3.11), we have shown the desired positive invariance of $\mathcal{R}(\alpha^\infty)$.

Proof of claim (3.11). Suppose that T^∞ is finite. Then, we have

$$|\theta_M(t)| < \alpha^\infty, \quad \text{for } t \in [0, T^\infty), \qquad |\theta_M(T^\infty)| = \alpha^\infty. \tag{3.12}$$

Now, we consider the rate of change of $|\theta_M|$ at time $t = T^\infty$:

$$
\begin{aligned}
\frac{d|\theta_M|}{dt}\Big|_{t=T^\infty} &= \Omega_M \mathrm{sgn}(\theta_M(T^\infty)) + \frac{K}{N} S(|\theta_M(T^\infty)|) \sum_{j=1}^N I(|\theta_j(T^\infty)|) \\
&\le \Omega^\infty + KS(|\theta_M(T^\infty)|) I(|\theta_M(T^\infty)|) \\
&= \Omega^\infty + KS(\alpha^\infty) I(\alpha^\infty) \quad \text{by (3.12)} \\
&= -S(\alpha^\infty) I(\alpha^\infty)(K_e - K) < 0,
\end{aligned} \tag{3.13}
$$

where we used the condition (2.7):

$$
\begin{aligned}
&\Omega^\infty = -K_e S(\alpha^\infty) I(\alpha^\infty), \quad S(|\theta_M(T^\infty)|) \le 0, \\
&0 < I(|\theta_M(T^\infty)|) \le I(|\theta_j(T^\infty)|), \quad 1 \le j \le N.
\end{aligned}
$$

The estimate (3.13) says that $|\theta_M|$ is in a strictly decreasing mode at $t = T^\infty$; thus, we have

$$|\theta_M(T^\infty - \delta_2)| > \alpha^\infty, \quad \text{for some } \delta_2 > 0.$$

This is contradictory to (3.12). Therefore, $T^\infty = \infty$, and we prove the positive invariance of $\mathcal{R}(\alpha^\infty)$. □

Remark 3.1. *Since the system* (1.1) *is autonomous, if the Winfree flow reaches the set* $\mathcal{R}(\alpha^\infty)$ *in any finite-time, then it will be trapped afterward, i.e., if* $\Theta(t_*) \in \mathcal{R}(\alpha^\infty)$ *for some finite* $t_* \in (0, \infty)$, *then* $\Theta(t) \in \mathcal{R}(\alpha^\infty)$, *for* $t > t_*$.

Proposition 3.1. (Transition to $\mathcal{R}(\alpha^\infty)$) *Suppose that the conditions* (2.6) *and* (2.7) *hold, and* α *and the coupling strength satisfy*

$$\alpha \in (0, \theta^*) \quad \textit{and} \quad K > K_e(\alpha^\infty).$$

Let $\Theta = (\theta_1, \cdots, \theta_N)$ *be a global smooth solution to* (1.1), *satisfying*

$$\Theta_0 \in \overline{\mathcal{R}(\alpha)}.$$

Then, there exists $t_* \ge 0$ *such that*

$$\Theta(t) \in \mathcal{R}(\alpha^\infty), \quad \textit{for } t > t_*.$$

Proof. As in Lemma 3.1, we introduce a maximal index M, such that

$$M := \mathrm{argmax}_{1 \le i \le N} |\theta_i|,$$

and note that $\Theta(t) \in \mathcal{R}(\alpha^\infty)$ is equivalent to $|\theta_M(t)| < \alpha^\infty$. For the proof, it is enough to show the existence of an entrance time t_* of θ_M into the interval $(-\alpha^\infty, \alpha^\infty)$, and then θ_M stays there afterwards as Remark 3.1.

• (Existence of an entrance time): We now show that $|\theta_M|$ enters $[0, \alpha^\infty)$ in some finite-time t_*.

⋄ Case A ($\alpha \le \theta_*$). In this case, we have $\alpha^\infty = \alpha$. We claim that $t_* = 0$. If $|\theta_M^0| < \alpha$, we are done. If $|\theta_M^0| = \alpha$, then we use the same argument as in (3.11) to find

$$\frac{d|\theta_M|}{dt}\Big|_{t=0} \le \max_{1 \le i \le N} |\Omega_i| + KS(\alpha) I(\alpha) = -S(\alpha) I(\alpha)(K_e - K) < 0.$$

⋄ Case B ($\theta_* < \alpha \le \theta^*$). We see that $|\theta_M| \in [\alpha^\infty, \alpha]$ implies

$$\frac{d|\theta_M|}{dt} \le -S(\alpha) I(\alpha)(K_e - K) < 0.$$

Hence, $|\theta_M|$ enters $[0, \alpha^\infty)$ in some time t_*:

$$t_* \geq \Big|\frac{\alpha - \alpha^\infty}{S(\alpha)I(\alpha)(K-K_e)}\Big|.$$

□

For $\alpha^\infty = \theta_*$, the argument of the next section does not hold. We can solve this problem by simply 'replacing' α^∞ with some appropriate β, as in the following lemma:

Lemma 3.2. *Suppose that the conditions* (2.6) *and* (2.7) *hold, and that* α *and the coupling strength satisfy*

$$\alpha \in (0, \theta^*) \quad \textit{and} \quad K > K_e(\alpha^\infty).$$

Also, let $\Theta = (\theta_1, \cdots, \theta_N)$ *be a global smooth solution to* (1.1)*, satisfying*

$$\Theta_0 \in \overline{\mathcal{R}(\theta_*)}.$$

Then, there exists $\beta \in (0, \theta_*)$ *and* $t_* \geq 0$*, such that*

$$\Theta(t) \in \mathcal{R}(\beta) \quad \textit{for} \quad t > t_*.$$

Proof. We can find an α' such that

$$\theta_* < \alpha' < \theta^*, \quad K > -\frac{\Omega^\infty}{S(\alpha')I(\alpha')}.$$

Then, it follows from Proposition 3.1 that there exists t_* such that

$$\Theta(t) \in \mathcal{R}((\alpha')^\infty) \quad \text{for } t > t_*.$$

We set $\beta := (\alpha')^\infty$ to derive the desired result. □

Remark 3.2. *In the remainder of this paper, we will assume that* $\alpha^\infty < \theta_*$*.*

3.2. **Exponential ℓ^1-stability.** In this subsection, we study the exponential ℓ^1-stability of the Winfree flow, with respect to initial data.

Proposition 3.2. *Suppose that the structural conditions* (2.6) *and* (2.7) *hold, and that for* $\alpha \in (0, \theta^*)$*, the initial data and coupling strength satisfy*

$$K > K_e \quad \textit{and} \quad \Theta_0^1,\ \Theta_0^2 \in \overline{\mathcal{R}(\alpha)}.$$

Let $\Theta^1 = (\theta_1^1, \cdots, \theta_N^1)$ *and* $\Theta^2 = (\theta_1^2, \cdots, \theta_N^2)$ *be two global smooth solutions to* (1.1)*, with corresponding to two initial configurations* Θ_0^1 *and* Θ_0^2*, respectively. Then, there exists* $t_* > 0$ *and negative constants* C_0, C_1*, such that*

$$\begin{aligned} e^{-K|C_0|(t-t_*)}\|\Theta^1(t_*) - \Theta^2(t_*)\|_1 &\leq \|\Theta^1(t) - \Theta^2(t)\|_1 \\ &\leq e^{-K|C_1|(t-t_*)}\|\Theta^1(t_*) - \Theta^2(t_*)\|_1, \quad t \geq t_*, \end{aligned}$$

where t_* *is as in Lemma 3.2, and the negative constants* C_0 *and* C_1 *are given as follows.*

$$C_0 := (S'I)(0) - (SI')(\alpha^\infty) < 0, \qquad C_1 := (SI)'(\alpha^\infty) < 0.$$

Proof. It follows from Lemma 3.2 that there exists $t_* \geq 0$, such that

$$\Theta^1(t), \Theta^2(t) \in \mathcal{R}(\alpha^\infty), \quad \text{for } t \geq t_*.$$

Then, we use (1.1) and Taylor's theorem applied to the function $(x, y) \mapsto S(x)I(y)$, to see that

$$
\begin{aligned}
&\frac{d}{dt}(\theta_i^1 - \theta_i^2) \\
&= \frac{K}{N}\sum_{j=1}^N \left[S(\theta_i^1)I(\theta_j^1) - S(\theta_i^2)I(\theta_j^2)\right] \\
&= \frac{K}{N}\left(\sum_{j=1}^N S'(\tilde\theta_i)I(\tilde\theta_j)\right)(\theta_i^1 - \theta_i^2) + \frac{K}{N}\sum_{j=1}^N S(\tilde\theta_i)I'(\tilde\theta_j)(\theta_j^1 - \theta_j^2), \quad t \ge t_*,
\end{aligned} \tag{3.14}
$$

for some $\tilde\theta_i \in (-\alpha^\infty, \alpha^\infty)$. Next, we multiply $\mathrm{sgn}(\theta_i^1 - \theta_i^2)$ with (3.14), and use the relation

$$
(\theta_j^1 - \theta_j^2) = \mathrm{sgn}(\theta_j^1 - \theta_j^2)|\theta_j^1 - \theta_j^2|,
$$

and we sum over i, to obtain

$$
\begin{aligned}
&\frac{d}{dt}\|\Theta^1(t) - \Theta^2(t)\|_1 \\
&= \frac{K}{N}\sum_{i=1}^N \underbrace{\Big[\sum_{j=1}^N S'(\tilde\theta_i)I(\tilde\theta_j)\Big]}_{=:\mathcal{I}_1(t)}|\theta_i^1 - \theta_i^2| \\
&\quad + \frac{K}{N}\sum_{j=1}^N \underbrace{\Big[\sum_{i=1}^N S(\tilde\theta_i)I'(\tilde\theta_j)\mathrm{sgn}(\theta_i^1 - {\theta^2}_i)\mathrm{sgn}(\theta_j^1 - \theta_j^2)\Big]}_{=:\mathcal{I}_2(t)}|\theta_j^1(t) - {\theta^2}_j(t)|, \ t \ge t_*.
\end{aligned} \tag{3.15}
$$

⋄ (Estimate of $\mathcal{I}_1$): We use (2.7), to see that

$$
S'(0) \le S'(\tilde\theta_i) \le S'(\alpha^\infty) < 0, \quad I(0) \ge I(\tilde\theta_j) \ge I(\alpha^\infty) > 0.
$$

This yields:

$$
NS'(0)I(0) \le \mathcal{I}_1(t) \le NS'(\alpha^\infty)I(\alpha^\infty) < 0, \quad t \ge t_*. \tag{3.16}
$$

⋄ (Estimate of $\mathcal{I}_2$): We also use (2.7), to see that

$$
S(\alpha^\infty) \le S(\tilde\theta_i) \le S(0) \le 0, \quad I'(\alpha^\infty) \le I'(\tilde\theta_j) \le I'(0) \le 0.
$$

This again yields:

$$
0 \le S(0)I'(0) \le S(\tilde\theta_i)I'(\tilde\theta_j) \le S(\alpha^\infty)I'(\alpha^\infty).
$$

Thus, we have

$$
-NS(\alpha^\infty)I'(\alpha^\infty) \le \mathcal{I}_2(t) \le NS(\alpha^\infty)I'(\alpha^\infty). \tag{3.17}
$$

For the upper bound estimate, we use (3.15), (3.16), and (3.17), to derive

$$
\frac{d}{dt}\|\Theta^1(t) - \Theta^2(t)\|_1 \le K(SI)'(\alpha^\infty)\|\Theta^1(t) - \Theta^2(t)\|_1. \tag{3.18}
$$

Similarly, the lower bound estimate can be obtained as follows.

$$
\frac{d}{dt}\|\Theta^1(t) - \Theta^2(t)\|_1 \ge K\Big[(S'I)(0) - (SI')(\alpha^\infty)\Big]\|\Theta^1(t) - \Theta^2(t)\|_1. \tag{3.19}
$$

Finally, we combine (3.18) and (3.19) to obtain the desired estimates. □

Remark 3.3. *Note that $(SI)'(\alpha^\infty) < 0$. Thus, $\|\Theta^1(t) - \Theta^2(t)\|_1$ exponentially decays, with a decay rate lying in $[K((S'I)(0) - (SI')(\alpha^\infty)), K((SI)'(\alpha^\infty))]$, for $t > t_*$.*

4. Existence of unique equilibrium and its structure

In this section, we show the existence of an equilibrium solution to (1.1), and provide its structure. Throughout the paper, we assume that the conditions (2.6) and (2.7) hold, and

$$\alpha \in (0, \theta^*), \quad K > K_e.$$

Let $\Phi = (\phi_1, \dots, \phi_N) \in \overline{\mathcal{R}(\alpha)}$ be an equilibrium solution to (1.1). Then, it satisfies the equilibrium system:

$$\Omega_i + \frac{K}{N} S(\phi_i) \sum_{j=1}^{N} I(\phi_j) = 0, \quad i = 1, \cdots, N. \tag{4.20}$$

The system (4.20) can be rewritten into two cases as

$$\begin{aligned} &\frac{K}{N} S(\phi_i) \sum_{j=1}^{N} I(\phi_j) = 0, \quad \text{for } \Omega_i = 0, \\ &1 + \frac{K}{N} \frac{S(\phi_i)}{\Omega_i} \sum_{j=1}^{N} I(\phi_j) = 0, \quad \text{for } \Omega_i \neq 0. \end{aligned} \tag{4.21}$$

Now, Proposition 3.1 implies that Φ must lie in $\mathcal{R}(\alpha^\infty)$. Furthermore, Proposition 3.2 implies that any $\Theta(t)$ satisfying (1.1) converges to Φ, which also furnishes the uniqueness of Φ up to given α and Ω^∞. We now define a continuous function

$$F(s) := 1 + \frac{K}{N} s \sum_j I(S^{-1}(\Omega_j s)) = 0.$$

Lemma 4.1. *There exists a root $\tilde{s}$ of $F(s) = 0$ in the interval $\left[\frac{S(\alpha^\infty)}{\Omega^\infty}, 0\right]$.*

Proof. Note that the sensitivity function S is strictly decreasing on the interval $[-\theta_*, \theta_*]$. To prove the existence of a root, we will use the intermediate value theorem. First, note that F is continuous on $[\frac{S(\alpha^\infty)}{\Omega^\infty}, 0]$, and $F(0) = 1 > 0$. We will show that $F\left(\frac{S(\alpha^\infty)}{\Omega^\infty}\right) < 0$.

• ($\Omega_j \geq 0$): Suppose that $\Omega_j \geq 0$, for some $j \in \{1, \cdots, N\}$. Since $S(\alpha^\infty) < 0$, we have that

$$S(\alpha^\infty) \leq \frac{\Omega_j}{\Omega^\infty} S(\alpha^\infty) \leq 0.$$

Since S^{-1} is decreasing on $(S(\theta_*), S(-\theta_*))$, we have that

$$\alpha^\infty = S^{-1}\left(S(\alpha^\infty)\right) \geq S^{-1}\left(\frac{\Omega_j}{\Omega^\infty} S(\alpha^\infty)\right) > 0$$

We can now use the fact I is decreasing on $(0, \theta_*)$, to obtain

$$I(\alpha^\infty) \leq I\left(S^{-1}\left(\frac{\Omega_j}{\Omega^\infty} S(\alpha^\infty)\right)\right). \tag{4.22}$$

• ($\Omega_j < 0$): Suppose that $\Omega_j < 0$, for some $j \in \{1, \cdots, N\}$. In this case, we have that

$$0 < \frac{\Omega_j}{\Omega^\infty} S(\alpha^\infty) \leq -S(\alpha^\infty) = S(-\alpha^\infty),$$

because S is an odd function. Since S^{-1} is decreasing on $(S(\theta_*), S(-\theta_*))$, we have that

$$0 > S^{-1}\left(\frac{\Omega_j}{\Omega^\infty}S(\alpha^\infty)\right) \geq S^{-1}\left(S(-\alpha^\infty)\right) = -\alpha^\infty.$$

By using the fact that I is a even function, and increasing on $(-\theta_*, 0)$, we obtain

$$I(\alpha^\infty) = I(-\alpha^\infty) \leq I\left(S^{-1}\left(\frac{\Omega_j}{\Omega^\infty}S(\alpha^\infty)\right)\right). \tag{4.23}$$

We now combine (4.22) and (4.23), to derive the following inequality:

$$\begin{aligned} F\left[\frac{S(\alpha^\infty)}{\Omega^\infty}\right] &= 1 + \frac{K}{N}\frac{S(\alpha^\infty)}{\Omega^\infty}\sum_{j=1}^N I\left(S^{-1}\left(\Omega_j\frac{S(\alpha^\infty)}{\Omega^\infty}\right)\right) \\ &\leq 1 + \frac{K}{N}\frac{S(\alpha^\infty)}{\Omega^\infty}\sum_{j=1}^N I\left(\alpha^\infty\right) \\ &= 1 + \frac{KS(\alpha^\infty)I(\alpha^\infty)}{\Omega^\infty} < 0, \quad \text{for } K > K_e(\alpha^\infty). \end{aligned}$$

Thus, there exists a root $\tilde{s}$ of the equation $F(s) = 0$, by the intermediate value theorem. □

4.1. **Existence of a unique equilibrium in $\mathcal{R}(\alpha^\infty)$.** In this part, we will show the unique existence of phase-locked state in $\mathcal{R}(\alpha^\infty)$.

• (Existence): For the existence part, we split the construction into two cases.

◇ Case A ($\Omega_i = 0,\ i = 1, \cdots, N$). In this case, we set

$$\phi_i = 0, \quad i = 1, \cdots, N, \quad \text{i.e.,} \quad \Phi = 0.$$

Since $S = S(\theta)$ is an odd function, $S(0) = 0$. Thus, $\Phi = 0$ is the desired equilibrium.

◇ Case B ($\max_{1\leq i\leq N}|\Omega_i| > 0$). Note that the sensitivity function S is strictly decreasing on the interval $[-\theta_*, \theta_*]$. Let $\tilde{s}$ be a root of $F(s) = 0$, on the interval $\left[\frac{S(\alpha^\infty)}{\Omega^\infty}, 0\right]$, i.e.,

$$1 + \frac{K}{N}\tilde{s}\sum_j I(S^{-1}(\Omega_j\tilde{s})) = 0, \tag{4.24}$$

and we set

$$\phi_i := S^{-1}(\Omega_i\tilde{s}) \qquad i = 1, \cdots, N, \qquad \Phi := (\phi_1, \cdots, \phi_N). \tag{4.25}$$

Since S^{-1} is an odd function, we note that $\phi_j = 0$ for $j \in \{i \in \{1, \cdots, N\} : \Omega_i = 0\}$. We next show that the Φ defined by the relation (4.25) satisfies the equilibrium system, i.e., it is an equilibrium solution. We substitute the relation (4.25) into (4.24), to find the relation (4.21):

$$0 = 1 + \frac{K}{N}\tilde{s}\sum_j I(S^{-1}(\Omega_j\tilde{s})) = 1 + \frac{K}{N}\frac{S(\phi_i)}{\Omega_i}\sum_j I(\phi_j),$$

where we used $\Omega_i\tilde{s} = S(\phi_i)$. Thus, the state Φ defined by (4.25) is an equilibrium state.

• (Uniqueness): Let Φ^1 and Φ^2 be two equilibria of (1.1). Then, it follows from Proposition 3.2 that

$$\lim_{t\to\infty} ||\Phi^1(t) - \Phi^2(t)||_1 = ||\Phi^1(t) - \Phi^2(t)||_1 = 0.$$

That is, $\Phi^1 = \Phi^2$. Thus, there exists a unique equilibrium in $\mathcal{R}(\alpha^\infty)$.

(Remarks on the existence of an equilibrium). In order to show the convergence of a global solution of (1.1) toward the equilibrium, we can give an alternative simple argument as follows. Let $\Theta = \Theta(t)$ be a global solution, corresponding to initial data Θ_0, whose existence is guaranteed by the standard Cauchy-Lipschitz theory, due to the uniform boundedness of the R.H.S. of (1.1). We will show that $\Theta(t)$ is a Cauchy sequence in t, in the complete metric space $(\mathbb{R}^n, ||\cdot||_1)$.

Consider a time-shifted function $\Theta^h(t) := \Theta(t+h),\ h > 0$. Since (1.1) is autonomous, $\Theta^h(t)$ is a solution to (1.1), with initial data $\Theta^h(0) = \Theta(h)$. Then, we apply an exponential ℓ_1-stability in Proposition 3.2, for two solutions Θ and Θ^h: there exists $t_* > 0$ such that

$$\|\Theta(t) - \Theta^h(t)\|_1 = \|\Theta(t) - \Theta(t+h)\|_1 \le \|\Theta(t_*) - \Theta(t_*+h)\|_1 e^{-K|C_1|(t-t_*)} \quad t \ge t_*,$$

for some constant C_1. This implies that $\Theta(t)$ is a Cauchy sequence in $(\mathbb{R}^n, ||\cdot||_1)$. Therefore, there exists an equilibrium Φ, which is also a phase-locked state such that

$$\Phi := \lim_{t\to\infty} \Theta(t).$$

This verifies the existence of a phase-locked state, via the time-asymptotic approach. However, for the detailed structure estimate of phase-locked state, the existence proof given in the context of using the intermediate value theorem will be employed in next subsection.

4.2. **Structure of the unique equilibrium.** In this subsection, we show that the number of collisions between Winfree oscillators is finite under the framework described in Section 2.

Definition 4.1. *Let $\Theta = (\theta_1, \cdots, \theta_N)$ be a solution to (1.1), and the i-th and j-th oscillators θ_i and θ_j collide at time t_0, if and only if there exists $0 < \delta \ll 1$ such that*

$$\theta_i(t_0) = \theta_j(t_0), \quad \theta_i(t) \neq \theta_j(t) \quad t \in (t_0 - \delta, t_0).$$

We next show that the number of collisions in the Winfree flow is finite.

Lemma 4.2. *Suppose that the conditions (2.6) and (2.7) hold, and that α and the coupling strength satisfy*

$$\alpha \in (0, \theta^*) \quad \text{and} \quad K > K_e(\alpha^\infty),$$

and let $\Theta = \Theta(t)$ be a global smooth solution to the system (1.1), satisfying

$$\alpha \in (0, \theta^*), \quad \Theta_0 \in \overline{\mathcal{R}(\alpha)}.$$

Suppose that $\Omega_i > \Omega_j$. Then, we have

(i) If $\theta_{i0} > \theta_{j0}$, then oscillators i and j never collide.
(ii) If $\theta_{i0} < \theta_{j0}$, then oscillators i and j collide exactly once.

Proof. (i) Suppose that

$$\Omega_i > \Omega_j \quad \text{and} \quad \theta_{i0} > \theta_{j0}.$$

We define

$$\mathcal{T} := \{T : \theta_i(t) > \theta_j(t),\ t \in [0, T)\} \quad \text{and} \quad T^* := \sup \mathcal{T}.$$

Then, it is clear that $T^* > 0$, and $T^* \in \mathcal{T}$. Suppose that $T^* < \infty$. Then, we should have that

$$\theta_i(t) > \theta_j(t), \quad t \in [0, T^*), \qquad \theta_i(T^*) = \theta_j(T^*). \tag{4.26}$$

This implies that $\theta_i - \theta_j$ is in the strictly increasing mode at time $t = T^*$:

$$\frac{d}{dt}\Big|_{t=T^*}(\theta_i - \theta_j) = \Omega_i - \Omega_j > 0.$$

Thus, there exists $1 \gg \delta > 0$ such that

$$(\theta_i - \theta_j)(T^* - \delta) < 0, \quad \text{i.e.,} \quad \theta_i(T^* - \delta) < \theta_j(T^* - \delta),$$

which is contradictory to (4.26). Thus, $T^\infty = \infty$, and we have the desired result.
(ii) Suppose that

$$\Omega_i > \Omega_j \quad \text{and} \quad \theta_{i0} < \theta_{j0}.$$

It follows from Theorem 2.1 that

$$\lim_{t\to\infty} \theta_i(t) = \phi_i, \quad \lim_{t\to\infty} \theta_j(t) = \phi_j.$$

On the other hand, since $\Omega_i > \Omega_j$, and using the construction of the unique equilibrium, we have

$$\phi_i = S^{-1}(\Omega_i \tilde{s}) > S^{-1}(\Omega_j \tilde{s}) = \phi_j.$$

Thus, there must be at least one crossing between the i-th and j-th oscillators. However, after the first crossing between the i-th and j-th oscillators, we come down to the case in (i), and conclude that after the first crossing time, there will be no further crossing. This completes the proof of (ii). ☐

Remark 4.1. *1. If $\Omega_i = \Omega_j$, then the oscillators do not collide if $\theta_{i0} \neq \theta_{j0}$, and always agree with each other if $\theta_{i0} = \theta_{j0}$, by the uniqueness of the solution to the Winfree flow.*
2. For a given initial configuration $\Theta_0 = (\theta_{10}, \cdots, \theta_{N0})$, the total number of collisions between oscillators in the Winfree flow is given by the cardinality of the following set:

$$\{(i,j) \in \mathcal{N} \times \mathcal{N} \ : \ \Omega_i > \Omega_j, \quad \theta_{i0} \leq \theta_{j0}\}.$$

Proposition 4.1. *Suppose that the conditions* (2.6) *and* (2.7) *hold, and that α and the coupling strength satisfy*

$$\alpha \in (0, \theta^*) \quad \text{and} \quad K > K_e(\alpha^\infty),$$

and let $\Theta = \Theta(t)$ be the global smooth solution to the system (1.1)*, satisfying*

$$\alpha \in (0, \theta^*), \quad \Theta_0 \in \overline{\mathcal{R}(\alpha)}.$$

(1) If $\Omega_i \geq 0$, then we have that

$$-S^{-1}\left(\frac{\Omega_i}{KI(0)}\right) \leq \phi_i \leq -S^{-1}\left(\frac{\Omega_i}{KI(\alpha^\infty)}\right). \tag{4.27}$$

On the other hand, if $\Omega_i < 0$, then we have that

$$-S^{-1}\left(\frac{\Omega_i}{KI(\alpha^\infty)}\right) \leq \phi_i \leq -S^{-1}\left(\frac{\Omega_i}{KI(0)}\right).$$

(2) If $\Omega_i \geq \Omega_j$, then we have that

$$-\frac{\Omega_i - \Omega_j}{KS'(0)I(0)} \leq \phi_i - \phi_j \leq -\frac{\Omega_i - \Omega_j}{KS'(\alpha^\infty)I(\alpha^\infty)}. \tag{4.28}$$

Proof. (1) We consider two cases:

$$\text{Either} \quad \Omega_i \geq 0, \quad \text{or} \quad \Omega_i < 0.$$

• Case A ($\Omega_i \geq 0$). In this case, it follows from (4.21) that we have

$$\phi_i = S^{-1}(\Omega_i \tilde{s}), \qquad \tilde{s} := -\frac{N}{K\sum_{j=1}^N I(\phi_j)}. \tag{4.29}$$

Since $\phi \in \mathcal{R}(\alpha^\infty)$, $0 \leq I(\alpha^\infty) \leq I(\phi_j) \leq I(0)$, and $\Omega_i > 0$, we have

$$-\frac{1}{KI(\alpha^\infty)} \leq \tilde{s} \leq -\frac{1}{KI(0)}, \qquad -\frac{\Omega_i}{KI(\alpha^\infty)} \leq \Omega_i\tilde{s} \leq -\frac{\Omega_i}{KI(0)}. \tag{4.30}$$

Finally, we use the fact that S^{-1} is monotone decreasing, and the relation $\phi_i = S^{-1}(\Omega_i s)$, to give (4.27).

• Case B ($\Omega_i < 0$). In this case, we analogously obtain the following inequality.

$$-\frac{\Omega_i}{KI(0)} \leq \Omega_i\tilde{s} \leq -\frac{\Omega_i}{KI(\alpha^\infty)}.$$

By taking S^{-1}, we get the desired result.

(2) Suppose that $\Omega_i \geq \Omega_j$. Then, we use (4.29) and the mean value theorem, to obtain

$$\begin{aligned} \phi_i - \phi_j &= S^{-1}(\Omega_i\tilde{s}) - S^{-1}(\Omega_j\tilde{s}) \\ &= (S^{-1})'(\tilde{\theta})(\Omega_i - \Omega_j)\tilde{s}, \quad \text{for some } \tilde{\theta} \in (-\alpha^\infty, \alpha^\infty). \end{aligned} \tag{4.31}$$

On the other hand, it follows from the convexity of S on $[0, \alpha^\infty]$ that

$$\frac{1}{S'(\alpha^\infty)} \leq (S^{-1})'(\tilde{\theta}) \leq \frac{1}{S'(0)}. \tag{4.32}$$

Finally, we combine (4.30), (4.31), and (4.32), to derive the desired estimate (4.28). □

5. Numerical examples

In this section, we provide several numerical examples, in order to confirm the analytical results in Section 3 and 4. Throughout this section, we use the fourth order Runge-Kutta method, with time step $\triangle t = 0.01$ and $N = 20$ for all numerical simulations.

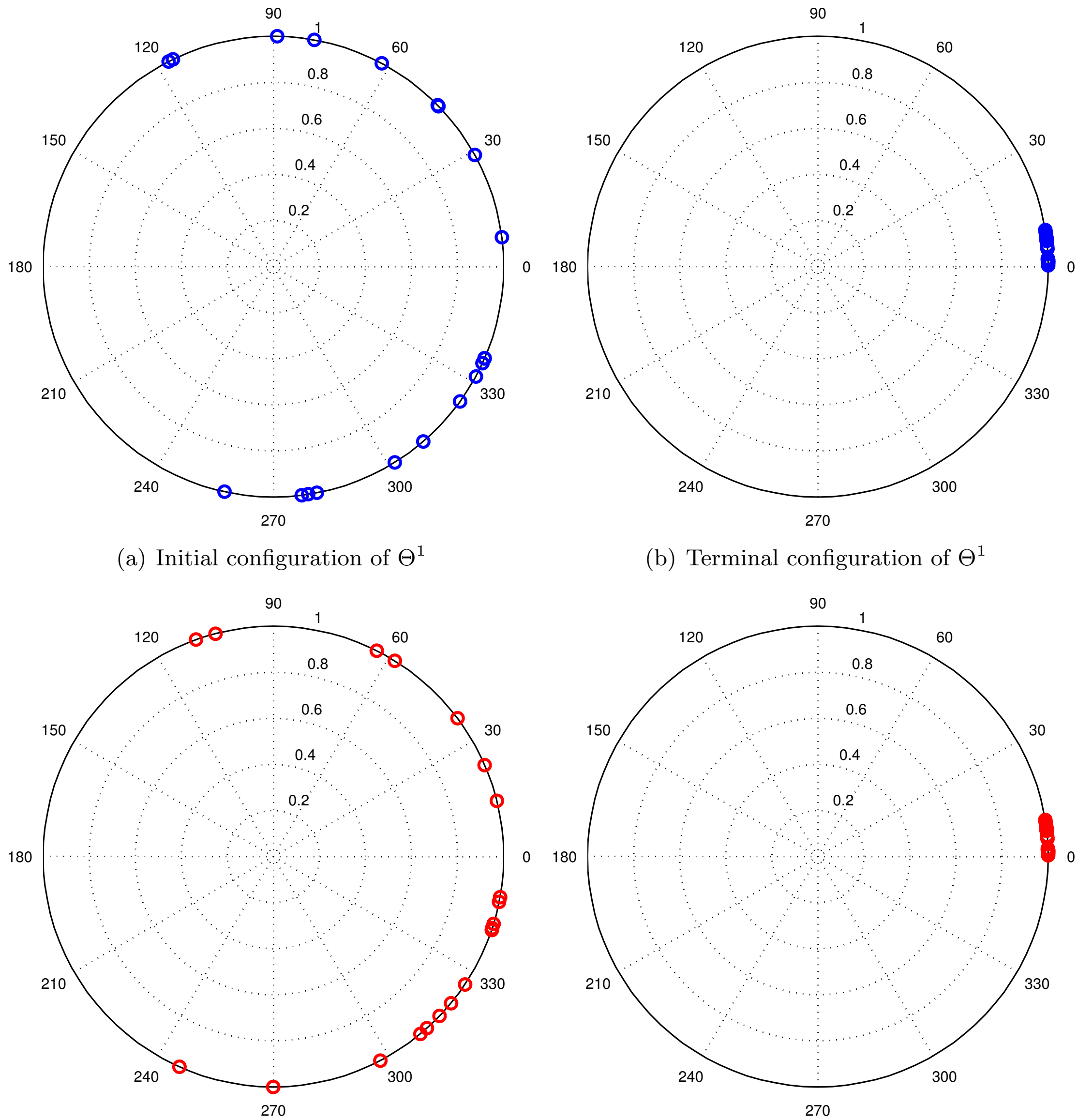
5.1. Emergence of a unique equilibrium.

To check the uniqueness of the phase-locked state under the framework given in Section 2.3., we employ two examples that have the same natural frequencies, but different initial configurations. We first explain our simulation set-up: we choose the special sensitivity function S and the influence function I, as follows.

$$S(\theta) = -\sin\theta, \qquad I(\theta) = 1 + \cos\theta.$$

As mentioned in Section 2.3., θ^* and θ_* can be taken as $\theta^* = \pi$ and $\theta_* = \frac{\pi}{3}$. We let $\alpha = \frac{2\pi}{3}$; then, $\alpha^\infty \approx 0.2210$ is determined by solving the equation (2.8). Natural frequencies are chosen randomly in $(0,1)$, and two different initial positions Θ_0^1 and Θ_0^2 are also chosen randomly in $\mathcal{R}(\alpha)$, as in Figures 3(a) and 3(c). In this setting, the lower bound for the coupling strength is given by

$$K_e(\alpha^\infty) := -\frac{\Omega^\infty}{S(\alpha^\infty)I(\alpha^\infty)} \approx 2.2229,$$



(a) Initial configuration of Θ^1

(b) Terminal configuration of Θ^1

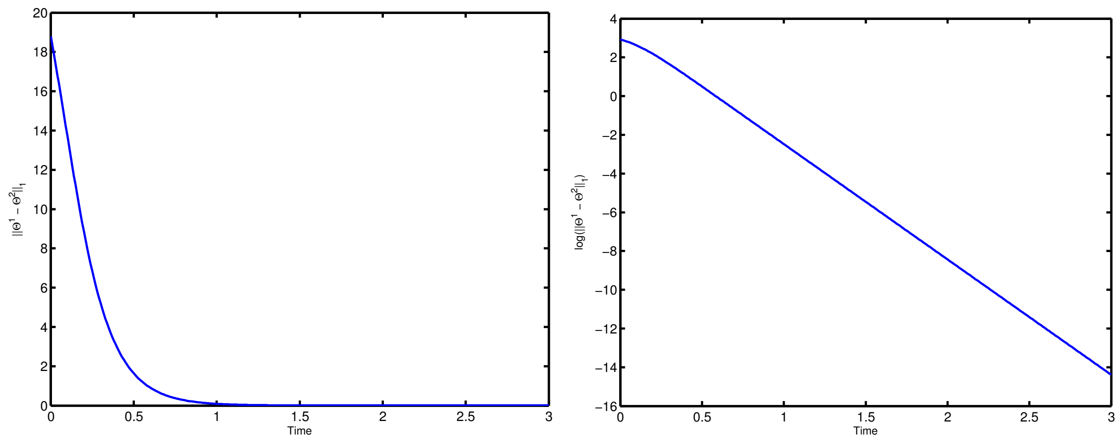
(c) Initial configuration of Θ^2

(d) Terminal configuration of Θ^2

FIGURE 3. Numerical simulations with two different initial positions

We set $K = 3$, so that $K > K_e$. We now compare two systems that have the same natural frequencies.

Figures 3(b) and 3(d) show that, although the initial positions are different, the solutions of each system converge to same terminal configuration, as expected by Theorem 2.1. In Figure 4, we plot the ℓ_1-difference of the two solutions. The linear decrease exhibited by $\log \|\Theta^1 - \Theta^2\|_1$ supports the exponential decay of the ℓ_1-difference. From Proposition 3.2,



(a) $\|\Theta^1 - \Theta^2\|_1$ (b) $\log \|\Theta^1 - \Theta^2\|_1$

FIGURE 4. ℓ_1 stability of Θ

We have that

$$\begin{aligned} KC_0(t - t_*) + \log \|\Theta^1(t_*) - \Theta^2(t_*)\|_1 &\le \log \|\Theta^1(t) - \Theta^2(t)\|_1 \\ &\le KC_1(t - t_*) + \log \|\Theta^1(t_*) - \Theta^2(t_*)\|_1, \end{aligned}$$

where $K = 3, C_0 = (S'I)(0) - (SI')(\alpha^\infty) \approx -2.0480$, and $C_1 = (SI)'(\alpha^\infty) \approx -1.8796$. Thus, the slope of $\log \|\Theta^1(t) - \Theta^2(t)\|_1$ should be between $KC_0 \approx -6.1441$ and $KC_1 \approx -5.6388$. If we calculate the average rate of change on the interval $(2, 3)$ in Figure 4(b), then

$$\log \|\Theta^1(3) - \Theta^2(3)\|_1 - \log \|\Theta^1(2) - \Theta^2(2)\|_1 \approx -5.9550,$$

which agrees with the analytic result given in Propositon 3.2.

5.2. **Exponential attracting property of the equilibrium.** In this part, we will examine the properties of the phase-locked state. Figure 5 shows the numerical results of the system with initial values $\Theta^1(0)$ in the previous setting. In Figure 5(b), we can see that the oscillators collide only a finite number of times, and settle into the phase-locked state.

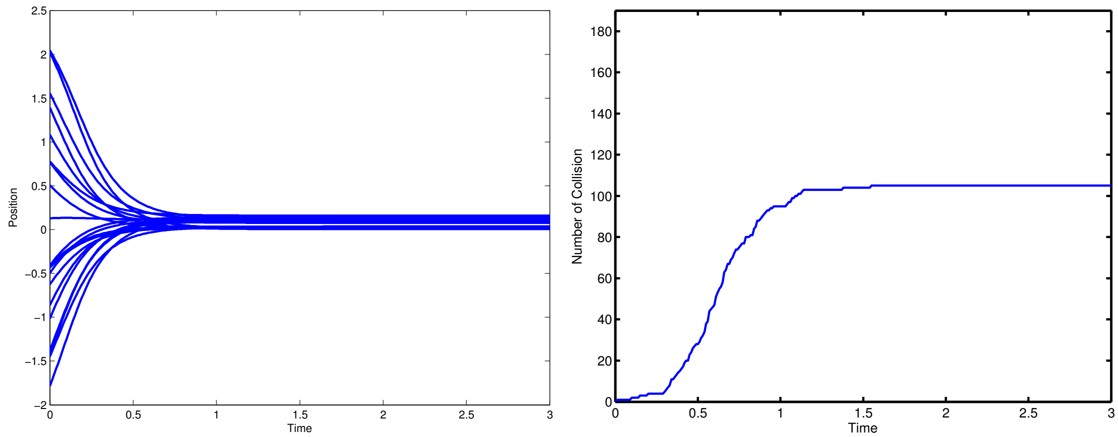
We now check the results from Propositon 4.1. We take ϕ_1 as an example, and compare with the upper and lower bounds given in (4.27),

$$-S^{-1}\left(\frac{\Omega_1}{KI(0)}\right) \le \phi_1 \le -S^{-1}\left(\frac{\Omega_1}{KI(\alpha^\infty)}\right).$$

In Figure 6(a), the red and green dotted lines are given upper and lower bounds, respectively. In (4.28), we can derive

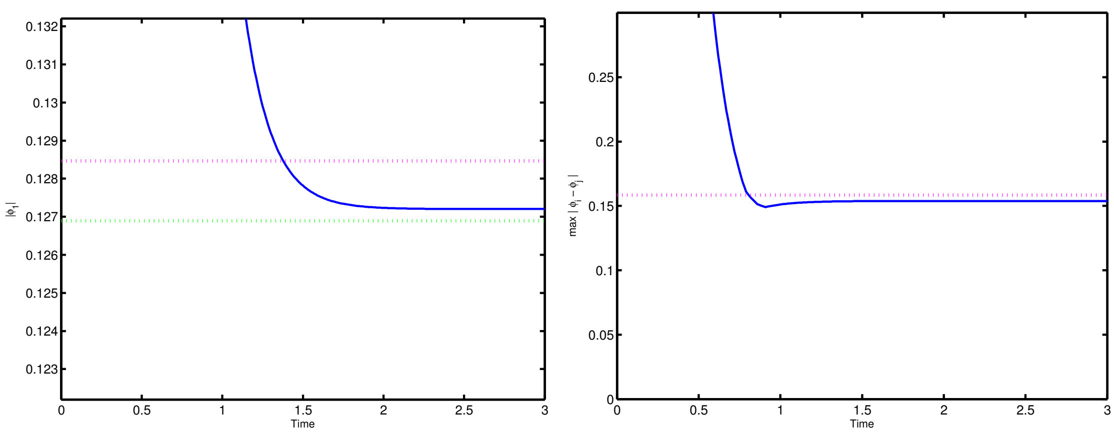
$$\max_{1 \le i,j \le N} |\phi_i - \phi_j| \le -\frac{\max_{i,j} |\Omega_i - \Omega_j|}{KS'(\alpha^\infty)I(\alpha^\infty)}.$$

In Figure 6(b), the red dotted line represents the given upper bound. Hence, the numerical simulations agree with the results in Proposition 4.1.



(a) Trajectories of θ_i (b) Number of collisions

Figure 5. Collision Estimate



(a) Upper and lower bounds of ϕ_1 (b) Diameter $\max_{i,j} |\phi_i - \phi_j|$

Figure 6. Estimate of ϕ_i

6. Conclusion

In this paper, we have studied an emergent phenomenon of the Winfree model, which has been proposed for the synchronization of weakly coupled limit-cycle oscillators. Unlike the well-studied Kuramoto phase model, the Winfree model lacks any conservation laws and symmetry in the coupling. Thus, the systematic approach from existing literature, based on the conservation laws and the Lyapunov functionals, seems to be difficult to apply directly. However, due to the structural conditions on the coupling functions, we have been able to present a sufficient framework for the emergence of unique phase-locked state, in terms of the sensitivity and influence functions comprising the coupling mechanism in the Winfree model. We showed that in a sufficiently large coupling regime, there exists a unique equilibrium for the Winfree model, and that it is asymptotically stable, so that it attracts all of its neighboring configurations exponentially fast.

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