SYNCHRONIZATION OF KURAMOTO OSCILLATORS WITH ADAPTIVE COUPLINGS

SEUNG-YEAL HA, SE EUN NOH, AND JINYEONG PARK

Abstract. We study the synchronization of Kuramoto oscillators with adaptive coupling in interacting networks. Network dynamics preserves the sum of all incoming pairwise coupling strengths and is designed to adaptively interact with system dynamics. For adaptive couplings, we use two adaptive coupling laws for the pairwise coupling strength. Kuramoto oscillators are assumed to be on the nodes of the networks. We present frameworks that guarantee the emergence of synchronization for various coupling feedback laws. Our results generalize earlier work on the synchronization of Kuramoto oscillators in fixed and symmetric networks.

1. Introduction

Collective phenomena in many biological complex systems are often observed and reported [1, 22, 26]. However, the systematic research of such phenomena is fairly recent; Kuramoto and Winfree [18, 19, 29] began systematically studying collective phenomena only fifty years ago. Since then, many phenomenological models for synchronization have been proposed in diverse scientific disciplines such as control theory, nonlinear dynamics, and statistical mechanics [1, 22]. We are primarily interested in the Kuramoto model, which is a first-order system of ordinary differential equations with sinusoidal couplings among phase oscillators. In the original Kuramoto model [18, 19], the coupling strength was assumed to be mean-field and constant. However, this constant coupling strength assumption is a bit too restrictive. It is more reasonable to vary the coupling strength based on the closeness of the initial configuration from the synchronized states; if the initial configuration is close to the synchronized state, a large coupling strength may be unnecessary, whereas if the configuration is far away from the synchronized state, a large coupling strength may be more appropriate. Thus, it is reasonable to introduce adaptive phase models for incorporating the coupling of the phase configuration and coupling strength. On the other hand, from a network perspective, the most synchronization models appearing in statistical mechanics and control theory a priori assume the global, static and all-to-all connectivity between oscillators e.g. [1, 9, 17, 18, 19, 20, 28]. However, for some biological complex systems exhibiting synchronous dynamics, network structure may develop according to system dynamics on it [21]. Thus, in this paper, we challenge the constant coupling strength assumption in

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the original Kuramoto model by allowing the coupling strengths to be dynamically wired, which is closer to real-world situations [6]. In fact, this adaptive coupling between phases and coupling strength was recently addressed in [6, 22, 23, 25] for biological and physical contexts. In social and biological networks, it is reasonable to assume that mutual coupling strengths are also dependent on the pair of oscillators and are time-dependent. Thus, it is more natural to include a dynamic law for the pairwise coupling strength that adapts to the phase configuration. In this paper, we study two self-consistent Kuramoto type models proposed in [2, 11, 21, 23, 24] (see Section 2.3 for a detailed discussion). Let $\theta_i = \theta_i(t)$ be the phase of the i-th oscillator with natural frequency ω_i, and let $k_{ij}(t)$ be the dynamic coupling strength between the i-th and j-th oscillators. In this situation, (θ_i, k_{ij}) is governed by the following Kuramoto model with adaptive coupling:

$$\begin{aligned} &\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i, j \le N, \\ &\dot{k}_{ij} = \mu \Gamma(\theta_j - \theta_i) - \gamma k_{ij}, \end{aligned} \tag{1.1}$$

subject to initial data:

$$\theta_i(0) = \theta_i^0 \quad \text{and} \quad k_{ij}(0) = k_{ij}^0. \tag{1.2}$$

Here, μ and γ are nonnegative constants proportional to learning enhancement rate and friction, respectively and Γ is a feedback law satisfying the parity and periodicity conditions,

$$\Gamma(-\theta) = \Gamma(\theta), \quad \Gamma(\theta + 2\pi) = \Gamma(\theta), \quad \theta \in \mathbb{R}. \tag{1.3}$$

Since the R.H.S. of (1.1) is 2π-periodic, (1.1) is a dynamical system on N-tori $\mathbb{S}^N$. However, we can lift the dynamical system as a dynamical system on the Euclidean space $\mathbb{R}^N$. Thus, the phase θ_i is assumed to take a value in $\mathbb{R}$ instead of taking mod 2π.

Note that system (1.1) cover the classical Kuramoto model for the special case:

$$\Gamma = 0, \quad \gamma = 0, \quad \text{and} \quad k_{ij}^0 = \frac{k}{N}, \quad 1 \le i, j \le N,$$

i.e., system (1.1) reduces to the globally coupled Kuramoto model [18, 19]:

$$\dot{\theta}_i = \omega_i + \frac{k}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i, j \le N.$$

It is well-known [1, 9] that the Kuramoto model exhibits a spontaneous phase transition from disordered states to ordered states as the coupling strength k increases from zero to infinity, and spontaneous synchronous dynamics emerge in the ensemble of Kuramoto oscillators. However, this plausible scenario has not been completely understood from a rigorous mathematical viewpoint, although there are several recent mathematical studies available [3, 4, 5, 7, 8, 9, 10, 12, 13, 14, 15, 17, 27].

The main results of this paper are as follows. We first consider evolution of coupling strengths with Hebbian-like adaptive coupling $\Gamma_c(\theta) = \cos\theta$:

$$\dot{k}_{ij} = \mu \cos(\theta_j - \theta_i) - \gamma k_{ij}, \quad t > 0. \tag{1.4}$$

Note that when the i-th oscillator and j-th oscillator are in phase, i.e., $\theta_j = \theta_i$, the gain term in the R.H.S. of (1.4) takes the maximum value. Thus, the choice of Γ_c is equivalent to the

Hebbian learning rule (see [16, 21, 24, 25]), whereas the i-th oscillator and j-th oscillator are in antiphase, the coupling strength k_{ij} is in a decreasing mode. In the sequel, we call Model A by the system (1.1) with an adaptive coupling Γ_c:

$$\text{Model A}: \quad \begin{cases} \dot{\theta}_i = \omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i, j \le N, \\ \dot{k}_{ij} = \mu \cos(\theta_j - \theta_i) - \gamma k_{ij}. \end{cases}$$

Our first two main results are to provide sufficient conditions leading to complete synchronization and practical synchronization in an asymptotic sense. More precisely, for identical oscillators with $\omega_i = \omega,\ 1 \le i \le N$, as long as the initial phase configuration is strictly confined to an arc with length $\frac{\pi}{2}$, we show that the phase diameter converges to zero at most exponentially as time progresses (Theorem 2.1). In contrast, for the non-identical oscillators case, we show that the phase diameter satisfies

$$\limsup_{t\to\infty} \Big(\max_{1\le i,j\le N} |\theta_i(t) - \theta_j(t)| \Big) \le \frac{\mathcal{O}(1)}{k_m^0}, \quad \text{where} \quad k_m^0 := \min_{i,j} k_{ij}^0.$$

Thus, as $k_m^0 \to \infty$, we have asymptotic practical synchronization (Theorem 2.2).

We next consider an adaptive scheme that for a pair of oscillators whose phase differences are small, we only need a small coupling strength to lock together, whereas for a pair of oscillators whose phase differences are big, we need a large coupling strength to lock together. To enhance the synchronization for the network of oscillators with different intrinsic frequencies, anti-Hebbian like coupling $\Gamma_s(\theta) := |\sin\theta|$ was designed in [2, 23], i.e.,

$$\dot{k}_{ij} = \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}, \quad t > 0. \tag{1.5}$$

Note that the coupling coefficient grows stronger for the pair of oscillators with larger phase difference, and the analysis of k_{ij} is more delicate than Model A, because the coupling strengths can approach zero as complete phase synchronization occurs for identical oscillators. Throughout the paper, we call Model B by the system (1.1) with an adaptive coupling law (1.5):

$$\text{Model B}: \quad \begin{cases} \dot{\theta}_i = \omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad 1 \le i, j \le N, \quad t > 0, \\ \dot{k}_{ij} = \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}, \end{cases}$$

Our main results for Model B can be summarized as follows. We first consider the identical oscillators. For a two-body system with the same natural frequency $\omega_1 = \omega_2$, it is important to estimate the asymptotic behavior of the ratio $\frac{|k_{12}|}{|\theta_1 - \theta_2|}$ (Lemma 4.1):

$$\lim_{t\to\infty} \frac{k_{12}(t)}{|\theta_1(t) - \theta_2(t)|} = \frac{\mu}{\gamma}.$$

Based on the above crucial estimate, for an initial configuration satisfying

$$|\theta_1^0 - \theta_2^0| < \pi, \qquad k_{12}^0 > 0,$$

we show that

$$|\theta_1(t) - \theta_2(t)| + k_{12}(t) \le \mathcal{O}(1)(1+t)^{-1} \quad \text{as } t \to \infty.$$

For a many-body system with $N \geq 3$, we use a Lyapunov functional approach to derive complete synchronization for identical oscillators (Theorem 2.3). More precisely, for an initial phase configuration Θ^0 with $\max_{1\leq i,j\leq N} |\theta_i^0 - \theta_j^0| < \frac{\pi}{2}$, we show that

$$\lim_{t\to\infty} \Big(\max_{1\leq i,j\leq N} k_{ij}(t) + \max_{1\leq i,j\leq N} |\dot{\theta}_i(t) - \dot{\theta}_j(t)| \Big) = 0.$$

Finally, for the non-identical oscillators, complete frequency synchronization is achieved for configurations with a diameter that is strictly less then $\frac{\pi}{2}$ in a priori setting (see Theorem 2.4).

The rest of this paper is organized as follows. In Section 2, we present several a priori estimates and introduce the main theorems with suitable frameworks. In Section 3, we study (1.1) with Hebbian-like adaptive function $\Gamma(\theta) = \cos\theta$ and provide complete and practical synchronization for identical and non-identical oscillators. In Section 4, we consider anti Hebbian-like adpative function, $\Gamma(\theta) = |\sin\theta|$, for a two-particle system. In Section 5, we generalize the results of Section 4 to a many-body system with $N \geq 3$. Finally, Section 6 is devoted to the summary of our main results.

Notation: For $\Theta = (\theta_1, \cdots, \theta_N),\ \Omega = (\omega_1, \cdots, \omega_N)$ and $K = [k_{ij}]$, we set

$$\begin{aligned}
&\|\Theta\| := \Big(\sum_{i=1}^N |\theta_i|^2 \Big)^{\frac{1}{2}}, \quad \|\Omega\| := \Big(\sum_{i=1}^N |\omega_i|^2 \Big)^{\frac{1}{2}}, \quad D(\Omega) := \max_{1\leq i,j\leq N} |\omega_i - \omega_j|,\\
&D(\Theta(t)) := \max_{1\leq i,j\leq N} |\theta_i(t) - \theta_j(t)|, \quad D(\dot{\Theta}(t)) := \max_{1\leq i,j\leq N} |\dot{\theta}_i(t) - \dot{\theta}_j(t)|,\\
&k_m(t) := \min_{1\leq i,j\leq N} k_{ij}(t), \qquad k_M(t) := \max_{1\leq i,j\leq N} k_{ij}(t).
\end{aligned}$$

2. Preliminaries

In this section, we present elementary estimates that will be used in later sections. We also provide a brief summary of our frameworks and main results. First of all, we recall the definitions of two synchronization concepts, namely, complete synchronization and practical synchronization as follows.

Definition 2.1. *Let (Θ, K) be a solution to (1.1). Then we have the following solution concepts for synchronization:*

(1) The phase configuration $\Theta = \Theta(t)$ exhibits asymptotic complete synchronization (ACS) if and only if the following two conditions hold:

$$\sup_{t\geq 0} D(\Theta(t)) < \infty, \quad \lim_{t\to\infty} D(\dot{\Theta}(t)) = 0.$$

(2) The phase configuration $\Theta = \Theta(t)$ exhibits asymptotic practical synchronization (APS) if and only if the following condition holds:

$$\lim_{k_m^0\to\infty} \limsup_{t\to\infty} D(\Theta(t)) = 0.$$

2.1. **Elementary estimates.** In this subsection, we study general properties of a Kuramoto type model (1.1) with adaptive couplings. Below, we study the invariance of symmetry and componentwise nonnegativity in the coupling matrix K.

Lemma 2.1. *Let* $(\Theta, K) \in \mathbb{R}^N \times \mathbb{R}^{N^2}$ *be a solution to system* (1.1)–(1.3) *with nonnegative initial coupling strengths*

$$k_{ij}^0 = k_{ji}^0 \geq 0, \quad 1 \leq i, j \leq N.$$

Then, the coupling matrix $K = [k_{ij}]$ *is componentwise nonnegative and symmetric:*

$$k_{ij}(t) = k_{ji}(t) \geq 0, \quad 1 \leq i, j \leq N, \ t > 0.$$

Proof. (i) (Nonnegativity): Suppose that the coupling strength matrix K satisfies the following: for fixed $i, j \in \{1, \cdots, N\}$, suppose that there exists a positive time $t_* \geq 0$ such that

$$k_{ij}(t_*) = 0.$$

It follows from (1.1) that

$$\dot{k}_{ij}(t_*) = \Gamma(\theta_j(t_*) - \theta_i(t_*)) - k_{ij}(t_*) = \Gamma(\theta_j(t_*) - \theta_i(t_*)) \geq 0.$$

Thus, k_{ij} is nondecreasing at $t = t_*$. Therefore, k_{ij} can not be negative for all t.

(ii) (Preservation of symmetry): Consider the difference $\Delta_{ij}(k) = k_{ij} - k_{ji}$. It is easy to verify that $\Delta_{ij}(k)$ satisfies

$$\frac{d}{dt}\Delta_{ij}(k) = -\gamma \Delta_{ij}(k), \quad t > 0, \qquad \Delta_{ij}(k)(0) = 0,$$

where the even parity of Γ is used. Thus, we have

$$\Delta_{ij}(k) = 0, \quad t > 0, \qquad \text{i.e.,} \qquad k_{ij}(t) = k_{ji}(t).$$

□

We next study the temporal evolution of total phase.

Lemma 2.2. *Let* (Θ, K) *be a solution to* (1.1) *with natural frequencies and the initial coupling strengths satisfying*

$$\sum_{i=1}^N \omega_i = 0 \quad \textit{and} \quad k_{ij}^0 = k_{ji}^0, \quad 1 \leq i, j \leq N.$$

Then, the total phase is conserved along the Kuramoto flow (1.1)–(1.3)*:*

$$\sum_{i=1}^N \theta_i(t) = \sum_{i=1}^N \theta_i^0, \qquad t \geq 0.$$

Proof. We first note that Lemma 2.1 implies

$$k_{ij} = k_{ji}, \quad 1 \leq i, j \leq N.$$

It follows from (1.1) and the odd parity of $k_{ij} \sin(\theta_j - \theta_i)$ in $i \leftrightarrow j$ that

$$\frac{d}{dt}\sum_{i=1}^N \theta_i = \sum_{i=1}^N \omega_i - \sum_{i,j=1}^N k_{ij} \sin(\theta_j - \theta_i) = \sum_{i=1}^N \omega_i = 0.$$

□

Without loss of generality, throughout the rest of this paper, we assume that the coupling strengths are symmetric, and the average of the natural frequencies is zero, i.e.,

$$k_{ij}(t) = k_{ji}(t), \quad \sum_{i=1}^{N} \omega_i = 0, \quad \text{and} \quad \sum_{i=1}^{N} \theta_i(t) = 0, \quad t \geq 0. \tag{2.1}$$

For a given phase configuration Θ, we set extremal indices M and m as follows:

$$\theta_M := \max_{1 \leq i \leq N} \theta_i, \qquad \theta_m := \min_{1 \leq i \leq N} \theta_i, \qquad D(\Theta) := \theta_M - \theta_m.$$

We next show that there exists a trapping set for the flow (1.1).

Lemma 2.3. (Existence of a trapping set) *Suppose that the natural frequency vector* Ω *and initial phase configuration* Θ^0 *satisfy*

$$\Omega = (\omega_1, \cdots, \omega_N) = 0, \qquad D(\Theta^0) < \pi.$$

Then for any dynamic solution Θ *of* (1.1)*,*

$$D(\Theta(t)) \leq D(\Theta^0), \quad t \geq 0.$$

Proof. First, set

$$\mathcal{T} := \{t \in [0, \infty) \ : \ D(\Theta(t)) < \pi\}, \quad T^\infty := \sup \mathcal{T}.$$

Then, since $D(\Theta^0) < \pi$ and by the continuity of $D(\Theta(t))$ with respect to t, there exists a $\delta > 0$ such that

$$[0, \delta) \subset \mathcal{T}.$$

We now claim that

$$T^\infty = \infty.$$

Proof of claim: Suppose $T^\infty < \infty$. Then on the finite interval $[0, T^\infty)$, it is easy to see from the dynamics of θ_i that the extremal phases θ_M and θ_m are nonincreasing and nondecreasing, respectively. More precisely,

$$\begin{aligned} \dot{\theta}_M &= \frac{1}{N} \sum_{j=1}^{N} k_{Mj} \sin(\theta_j - \theta_M) \leq 0, \quad t \in [0, T^\infty), \\ \dot{\theta}_m &= \frac{1}{N} \sum_{j=1}^{N} k_{mj} \sin(\theta_j - \theta_m) \geq 0, \quad t \in [0, T^\infty), \end{aligned}$$

where we use the fact that

$$-\pi < \theta_j - \theta_M \leq 0, \quad 0 \leq \theta_j - \theta_m < \pi.$$

Thus, we have

$$D(\Theta(t)) \leq D(\Theta^0) < \pi, \quad t \in [0, T^\infty).$$

Taking the left limit as $t \to T^\infty$ yields

$$D(\Theta(T^\infty)) \leq D(\Theta^0) < \pi.$$

Thus, $T^\infty \in \mathcal{T}$, and by the continuity of $D(\Theta(\cdot))$, there exists a $\delta' > 0$ such that

$$T^\infty + \delta' \in \mathcal{T}.$$

This contradicts the fact that T^∞ is the supremum of the set $\mathcal{T}$. Hence,

$$T^\infty = \infty, \quad \text{and} \quad D(\Theta(t)) \leq D(\Theta^0), \quad t \geq 0.$$

□

Finally, we present an estimate saying that as long as phases are confined in the half circle, then maximal phases are confined in some interval asymptotically.

Lemma 2.4. *For $T \in (0, \infty)$, let (Θ, K) be a solution to the coupled system* (1.1)–(1.3) *satisfying a priori assumption on the phase diameter*

$$\sup_{0 \le t < T} D(\Theta(t)) \le D_0 < \pi.$$

Then,we have

$$\|\Theta(t)\| \le \frac{\|\Omega\|}{NR_0 k_m} + \Big(\|\Theta^0\| - \frac{\|\Omega\|}{NR_0 k_m}\Big) e^{-NR_0 k_m t}, \quad 0 \le t < T,$$

where $R_0 := \frac{\sin D_0}{D_0}$

Proof. Multiplying (1.1) by $2\theta_i$, summing over i, and using the symmetry $k_{ij} = k_{ji}$ yields

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^N \theta_i^2 &= 2\sum_i \omega_i\theta_i + 2\sum_{i,j=1}^N k_{ij}\theta_i \sin(\theta_j - \theta_i) \\ &= 2\sum_i \omega_i\theta_i - 2\sum_{i,j=1}^N k_{ij}\theta_j \sin(\theta_j - \theta_i) \\ &= 2\sum_i \omega_i\theta_i - 2\sum_{i,j=1}^N k_{ij}(\theta_j - \theta_i)\sin(\theta_j - \theta_i). \end{aligned} \tag{2.2}$$

To derive the desired upper bound, we use

$$\begin{aligned} &(\theta_j - \theta_i)\sin(\theta_j - \theta_i) \ge R_0(\theta_j - \theta_i)^2 \quad \text{and} \\ &|\sin(\theta_i - \theta_j)|^2 \le |\theta_i - \theta_j|^2 \le (|\theta_i| + |\theta_j|)^2 \le 2(|\theta_i|^2 + |\theta_j|^2) \le 2\|\Theta\|^2. \end{aligned}$$

Thus, it follows from (2.2) that

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^N \theta_i^2 &\le 2\sum_i \omega_i\theta_i - R_0\sum_{i,j=1}^N k_{ij}|\theta_j - \theta_i|^2 \\ &\le 2\|\Omega\|\|\Theta\| - R_0 k_m \sum_{i,j=1}^N |\theta_j - \theta_i|^2 \\ &= 2\|\Omega\|\|\Theta\| - 2NR_0 k_m \sum_{i=1}^N \theta_i^2 \\ &= 2\|\Omega\|\|\Theta\| - 2NR_0 k_m \|\Theta\|^2. \end{aligned}$$

This yields the desired estimate.

□

2.2. **Frameworks and main results.** In this subsection, we briefly summarize our main frameworks and results for two different adaptive coupling laws for the dynamics of k_{ij} with

$$\Gamma_c(\theta) = \cos\theta, \quad \text{or} \quad \Gamma_s(\theta) = |\sin\theta|, \quad \theta \in \mathbb{R}.$$

2.2.1. *An adaptive law* Γ_c. We first recall model A with Γ_c in [21, 24, 25]:

$$\text{Model A}: \quad \begin{cases} \dot{\theta}_i = \omega_i + \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i, j \le N, \\ \dot{k}_{ij} = \mu \cos(\theta_j - \theta_i) - \gamma k_{ij}. \end{cases} \tag{2.3}$$

Note that the gain part of the dynamics for k_{ij} has maximum and minimum values, when i-th and j-th oscillators are in phase and antiphase, respectively, i.e., the growth of pairwise coupling strength k_{ij} is enhanced when i-th and j-th oscillators are in phase which is consistent to the Hebbian theory of neuroscience [16]. Our first theorem deals with the exponential synchronization of model A for identical oscillators with the common natural frequency $\omega_i = 0$, when initial phases are confined to an arc with a length $\frac{\pi}{2}$.

Theorem 2.1. (Identical oscillators) *Suppose that the natural frequency vector* Ω *and initial data* Θ^0 *satisfy*

$$\omega_i = 0, \quad 1 \le i \le N, \quad D(\Theta^0) < \frac{\pi}{2}, \quad \mu > 0, \quad \gamma > 0.$$

Then, for any solution $\Theta = \Theta(t)$ *to* (2.3), *there exists a positive number* $t_1 > 0$ *such that*

$$(i) \ \|\Theta(t)\| \le \|\Theta(t_1)\| \exp\Big[-\frac{\mu \cos D(\Theta^0) \sin D(\Theta^0)}{2\gamma D(\Theta^0)} (t - t_1) \Big], \quad t \ge t_1.$$
$$(ii) \ \|\Theta(t)\| \ge \|\Theta(t_1)\| \exp\Big[-\frac{2\mu}{\gamma} (t - t_1) \Big], \quad t \ge t_1.$$

Remark 2.1. *1. Note that even for an adaptive coupling case, exponential synchronization occurs as in the classical Kuramoto model so that the network structure does not change the nature of fast relaxation toward the synchronized state.*

2. Note that we do not impose any restrictions on initial coupling strengths k_{ij}^0. *As will be seen in Lemma 3.1, the pairwise coupling strength will be positive after some positive time even if they are negative or zero initially.*

We next return to the nonidentical oscillators. In this case, unfortunately, we cannot prove the ACS in Definition 2.1. Instead, we show that system (2.3) satisfies a kind of weak synchronization, namely practical synchronization in the sense of Definition 2.1.

Theorem 2.2. (Nonidentical oscillators) *Suppose that the parameters* μ, γ *and initial phase configuration and coupling strength satisfy the following conditions.*

$$\begin{aligned} &(i) \ \frac{\mu}{\gamma} > k_m^0 := \min_{i,j} k_{ij}^0 > 0, \quad D^\infty = \arccos\Big(\frac{\gamma k_m^0}{\mu}\Big) \in \Big(0, \frac{\pi}{2}\Big). \\ &(ii) \ D(\Theta^0) < D^\infty, \quad \sum_{i=1}^N \theta_i^0 = 0, \quad \sum_{i=1}^N \omega_i = 0, \quad 0 < D(\Omega) < \frac{k_m^0 N D(\Theta^0) \sin D^\infty}{D^\infty}. \end{aligned} \tag{2.4}$$

Then, for any solution $\Theta = \Theta(t)$ *to* (2.3), *practical synchronization is achieved. More precisely, we have*

$$\limsup_{t \to \infty} D(\Theta(t)) \le \frac{\sqrt{2} \|\Omega\| D^\infty}{N k_m^0 \sin D^\infty}$$

2.2.2. *An adaptive law* Γ_s. We next consider an adaptive coupling law Γ_s in [2, 23]. In this case, system (1.1) with Γ_s becomes

$$\text{Model B}: \quad \begin{cases} \dot{\theta}_i = \omega_i + \displaystyle\sum_{j=1}^{N} k_{ij}\sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i, j \le N, \\ \dot{k}_{ij} = \mu|\sin(\theta_j - \theta_i)| - \gamma k_{ij}. \end{cases} \tag{2.5}$$

Compared to Model A, the coupling term $|\sin(\theta_j - \theta_i)| \approx |\theta_j - \theta_i|$ assumes small values when $|\theta_j - \theta_i|$ so that the growth of coupling strength k_{ij} is suppressed as phase synchronization occurs. Thus, the synchronization estimates are very delicate compared to those of Model A. Even so, we can use Lyapunov type functional approach to derive asymptotic synchronization estimates for identical and non-identical oscillator systems.

Theorem 2.3. (Identical oscillators) *Let* $\Theta = \Theta(t)$ *be the global smooth solution to* (2.5) *satisfying*

$$\Omega = 0 \in \mathbb{R}^N \quad \textit{and} \quad D(\Theta^0) < \frac{\pi}{2}.$$

Then, we have ACS.

$$\lim_{t\to\infty} D(\Theta(t)) = 0, \qquad \lim_{t\to\infty} D(\dot{\Theta}(t)) = 0, \qquad \lim_{t\to\infty} \max_{1\le i,j\le N} |k_{ij}(t)| = 0.$$

and

Theorem 2.4. (Nonidentical oscillators) *Let* $\Theta = \Theta(t)$ *be a solution to* (2.5) *with the a priori assumption*

$$\sup_{0\le t<\infty} D(\Theta(t)) < \frac{\pi}{2}.$$

Then, we have ACS.

$$\lim_{t\to\infty} |D(\dot{\Theta})(t)| = 0, \quad 1 \le i, j \le N.$$

2.3. **Discussion on related works.** In this subsection, we briefly discuss some relevant literature on the co-evolving dynamics with adpative laws Γ_c and Γ_s in a weight network of phase oscillators. We first discuss phase coupled models with Γ_c type adaptive rules. In [21], Niyogi and English introduced a phase model with Hebbian learning:

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j\in N_i} k_{ij}\sin(\theta_j - \theta_i), \qquad \dot{k}_{ij} = \varepsilon\Big(\alpha\cos(\theta_i - \theta_j) - k_{ij}\Big),$$

where N_i is the neighbor set of i, and using numerical simulations, they studied the mutual effects of spontaneous synchronization and Hebbian learning in neuronal network. Motivated by neurophysiological models, Scardovi [24] introduced a model:

$$\dot{\theta}_i = \omega_i + \frac{1}{N}\sum_{j=1}^{N} k_{ij} f(\theta_j - \theta_i), \qquad \dot{k}_{ij} = \frac{1}{N}\Big(g(\theta_j - \theta_i) - k_{ij}\Big),$$

and showed that under suitable conditions on f, g and $k_{ij}(0)$, the above system can be written as a gradient system, and discuss a possible application to swarming systems. As a special case, we can take

$$f(\theta) = \sin\theta, \quad g(\theta) = \cos\theta, \quad k_{ij}(0) = k_{ji}(0), \quad 1 \le i, j \le N.$$

Next, we discuss the phase model with adaptive rule Γ_s. In [2], Aoki and Aoyagi proposed a co-evolving phase model with frustrations:

$$\dot{\theta}_i = 1 - \frac{1}{N}\sum_{j=1}^{N} k_{ij}\sin(\theta_i - \theta_j + \alpha), \qquad \dot{k}_{ij} = -\varepsilon \sin(\theta_i - \theta_j + \beta), \qquad |k_{ij}| \le 1,$$

where ε is the positive constant, and α, β are the interaction frustrations. Depending on the nature of the evolution of the coupling weight, the above system can exhibit three types of dynamical behavior: a two-cluster state, a coherent state with a fixed phase relation and a chaotic state with frustration. These observations have been studied analytically for a two-oscillator system and numerically for a many-oscillator system. On the other hand, Ren and Zhao [23] considered the phase model:

$$\dot{\theta}_i = \Omega_i - \frac{1}{N}\sum_{j=1}^{N} k_{ij}\sin(\theta_i - \theta_j), \qquad \dot{k}_{ij} = \varepsilon(|\sin(\beta(\theta_i - \theta_j))| - k_{ij}).$$

Based on numerical simulations, authors shows that the system dynamics approaches to the optimal coupling scheme in the sense of least average coupling cost in all-to-all couplings and nearest neighbor ring topology.

As presented in Section 2.2, our main results deal with explicit sufficient conditions on the initial phase and coupling strength leading to asymptotic complete synchronization or asymptotic practical synchronization for the Kuramoto phase model with adaptive coupling rules Γ_c and Γ_s and relaxation process such as decay rate toward the phase-locked states. Thus, our results are clearly different from [2, 21, 23, 24].

In the following two sections, we study synchronization estimates for Model A and Model B.

3. Synchronization estimates for Model A

In this section, we study the synchronization problem for Model A:

$$\begin{aligned} &\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} k_{ij}\sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i, j \le N, \\ &\dot{k}_{ij} = \mu\cos(\theta_j - \theta_i) - \gamma k_{ij}. \end{aligned} \tag{3.1}$$

For this, we first consider the simplest case where only two oscillators are present, and then discuss the many-oscillator case. Then, this yields the desired practical synchronization estimate. As a motivation, we first consider a two-oscillator system in the next subsection.

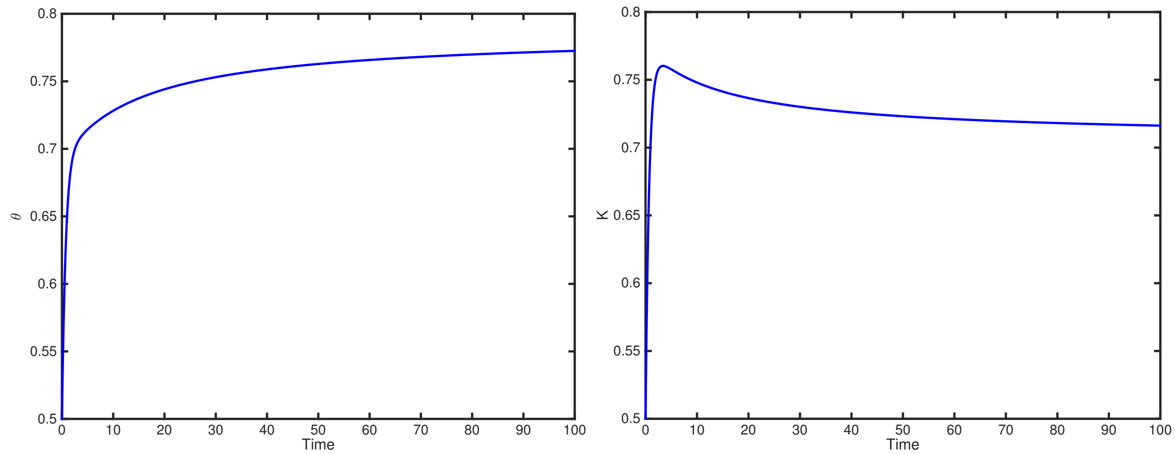
3.1. **A two-oscillator system.**

Consider a two-oscillator system for (2.3):

$$\begin{aligned} \dot{\theta}_1 &= \omega_1 + k_{12}\sin(\theta_2 - \theta_1), \quad t > 0, \\ \dot{\theta}_2 &= \omega_2 + k_{12}\sin(\theta_1 - \theta_2), \\ \dot{k}_{12} &= \mu\cos(\theta_2 - \theta_1) - \gamma k_{12}. \end{aligned} \tag{3.2}$$

To reduce these equations, we set

$$k := k_{12}, \quad \omega := \omega_2 - \omega_1, \quad \theta := \theta_2 - \theta_1.$$



(a) Dynamics of phase difference (b) Dynamics of coupling strength

FIGURE 1. Synchronization when $(\mu, \gamma, \omega) = (1, 1, 1)$ and $(\theta^0, k^0) = (0.5, 0.5)$.

In this case, (3.2) is reduced to

$$\dot{\theta} = \omega - 2k \sin\theta, \qquad \dot{k} = \mu \cos\theta - \gamma k, \quad t > 0. \tag{3.3}$$

Note that the second equation in (3.3) can be rewritten as

$$k(t) = k^0 e^{-\gamma t} + \mu \int_0^t e^{\gamma(s-t)} \cos\theta(s) ds. \tag{3.4}$$

We substitute (3.4) into the first equation of (3.3) to obtain an integro-differential equation:

$$\dot{\theta}(t) = \omega - 2k^0 e^{-\gamma t} \sin\theta(t) - 2\mu \Big[\int_0^t e^{\gamma(s-t)} \cos\theta(s) ds \Big] \sin\theta(t), \quad t \geq 0. \tag{3.5}$$

The numerical integrations for the dynamics of (θ, k) in (3.5) are shown in Figs. 1 and 2 for different parameter (μ, γ, ω). From the numerical simulations in Figs. 1 and 2, note that some restrictions on parameters must be imposed to guarantee ACS in the sense of Definition 2.1

3.2. **Many-oscillator system.** In this subsection, we consider a many-oscillator case with $N \geq 3$.

3.2.1. *Identical oscillators.* In this subsection, we present a synchronization estimate for (3.1) for identical oscillators ($\omega_i = \omega_j$). In this case, constraint (2.1) forces

$$\omega_i = 0, \quad 1 \leq i \leq N.$$

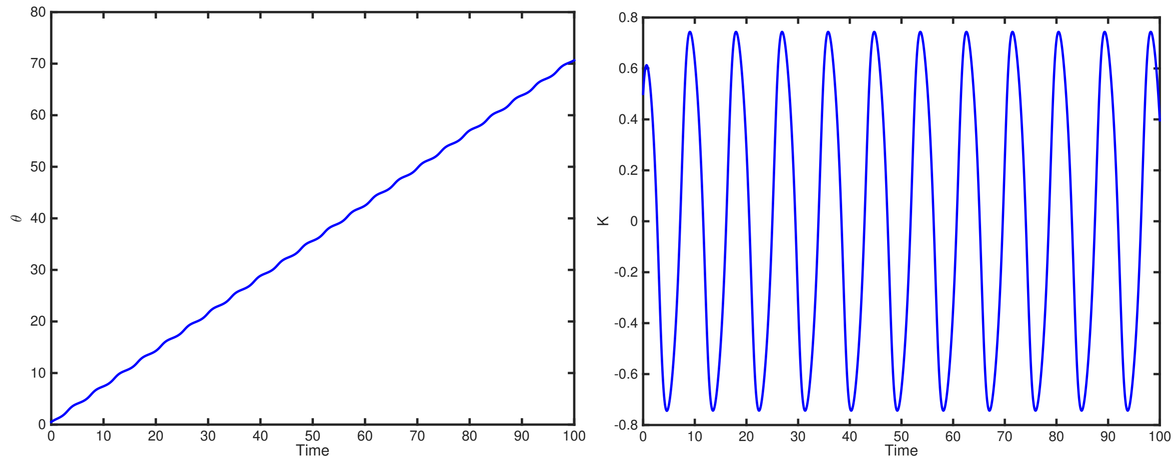
We first present the uniform boundedness of k_{ij}.

Lemma 3.1. *Suppose that the natural frequency vector Ω and initial data Θ^0 satisfy*

$$\Omega = 0 \in \mathbb{R}^N, \qquad D(\Theta^0) < \frac{\pi}{2},$$

and let Θ be a solution to (1.1), (1.2), and (2.1). Then, we have

$$\Big(k_{ij}^0 - \frac{\mu}{\gamma} \cos D(\Theta^0)\Big) e^{-t} + \frac{\mu}{\gamma} \cos D(\Theta^0) \leq k_{ij}(t) \leq \Big(k_{ij}^0 - \frac{\mu}{\gamma}\Big) e^{-\gamma t} + \frac{\mu}{\gamma}. \tag{3.6}$$



(a) Dynamics of phase difference (b) Dynamics of coupling strength

FIGURE 2. Desynchronization when $(\mu, \gamma, \omega) = (1, 1, 2)$ and $(\theta^0, k^0) = (0.5, 0.5)$.

Proof. It follows from Lemma 2.3 that

$$\sup_{0 \le t < \infty} D(\Theta(t)) \le D(\Theta^0) < \frac{\pi}{2}.$$

Thus, the feedback term Γ satisfies

$$\cos D(\Theta^0) \le \cos(\theta_j - \theta_i) \le 1.$$

Therefore, k_{ij} satisfies

$$\mu \cos D(\Theta^0) - \gamma k_{ij} \le \dot{k}_{ij}(t) \le \mu - \gamma k_{ij}.$$

Gronwall's lemma yields the desired results. □

Remark 3.1. *Estimate* (3.6) *implies*

$$\frac{\mu \cos D(\Theta^0)}{\gamma} \le \liminf_{t \to \infty} k_{ij}(t), \qquad \limsup_{t \to \infty} k_{ij}(t) \le \frac{\mu}{\gamma}.$$

We are now ready to provide the proof of our first result.

Proof of Theorem 2.1: Suppose that the initial data satisfy

$$D(\Theta^0) < \frac{\pi}{2}.$$

It follows from Lemma 2.3 that

$$\sup_{0 \le t < \infty} D(\Theta(t)) \le D(\Theta^0) < \frac{\pi}{2}.$$

Lemma 3.1 implies that there exists a positive number t_1 such that

$$\inf_{t \ge t_1} k_{ij}(t) \ge \frac{\mu}{2\gamma} \cos D(\Theta^0) \quad \text{and} \quad \sup_{t \ge t_1} k_{ij}(t) \le \frac{2\mu}{\gamma}. \tag{3.7}$$

• (Upper bound estimate): We use Lemma 2.4 and (3.7) to obtain

$$\frac{d}{dt}\|\Theta\| \le -\frac{\sin D(\Theta^0)}{D(\Theta^0)} k_m(t) \|\Theta\| \le -\frac{\sin D(\Theta^0)}{D(\Theta^0)} \frac{\mu}{2\gamma} \cos D(\Theta^0) \|\Theta\|, \quad t \ge t_1.$$

This yields

$$\|\Theta(t)\| \le \|\Theta(t_1)\| \exp\Big[-\frac{\mu\cos D(\Theta^0)\sin D(\Theta^0)}{2\gamma D(\Theta^0)}(t-t_1)\Big], \quad t \ge t_1.$$

• (Lower bound estimate): Again, we use Lemma 2.4 and (3.7) to obtain

$$\frac{d}{dt}\|\Theta\| \ge -k_M(t)\|\Theta\| \ge -\frac{2\mu}{\gamma}\|\Theta\|, \quad t \ge t_1.$$

Thus, we have

$$\|\Theta(t)\| \ge \|\Theta(t_1)\| \exp\Big[-\frac{2\mu}{\gamma}(t-t_1)\Big], \quad t \ge t_1.$$

This completes the proof of Theorem 2.1.

3.2.2. *Nonidentical oscillators.* In this part, we present a proof of Theorem 2.2. Our basic idea for synchronization estimates takes the following two steps.

- Step A (existence of a trapping set): We show that the pairwise coupling strength k_{ij} and phase-diameter $D(\Theta)$ satisfy

$$\inf_{0\le t<\infty} \min_{1\le i,j\le N} k_{ij} \ge k_m^0 > 0, \qquad \sup_{0\le t<\infty} D(\Theta(t)) < \frac{\pi}{2}.$$

- Step B (derivation of Gronwall's inequality): We derive a Gronwall type inequality for $D(\Theta)$:

$$\frac{d\|\Theta\|}{dt} \le \|\Omega\| - R^\infty k_m^0 \|\Theta\|.$$

First, we prove the existence of a trapping set.

Lemma 3.2. *Suppose that the parameters μ, γ and initial phase configuration and coupling strength satisfy the following conditions.*

(i) $\dfrac{\mu}{\gamma} > k_m^0 := \min\limits_{i,j} k_{ij}^0 > 0, \quad D^\infty = \arccos\Big(\dfrac{\gamma k_m^0}{\mu}\Big) \in \Big(0, \dfrac{\pi}{2}\Big).$

(ii) $D(\Theta^0) < D^\infty, \quad \sum\limits_{i=1}^N \theta_i^0 = 0, \quad \sum\limits_{i=1}^N \omega_i = 0, \quad 0 < D(\Omega) < \dfrac{k_m^0 N D(\Theta^0)\sin D^\infty}{D^\infty},$

and let Θ be a global solution to (3.1). *Then, we have*

$$\inf_{0\le t<\infty} k_m(t) \ge k_m^0, \qquad \sup_{0\le t<\infty} D(\Theta(t)) \le D^\infty.$$

Proof. The proof follows similarly to the proof of Lemma 3.2 in [12]. Thus, we briefly sketch the proof below. We first define the set $\mathcal{T}$ and its supremum:

$$\mathcal{T} := \{T \in [0,\infty) \;:\; D(\Theta(t)) < D^\infty, \quad t \in (0,T)\}, \qquad T^\infty := \sup \mathcal{T}.$$

Note that since $D(\Theta^0) < D^\infty$, and $D(\Theta(t))$ is a continuous function of t, there exists $\delta > 0$ such that

$$D(\Theta(t)) < D^\infty, \qquad \forall\; t \in [0,\delta).$$

Therefore, the set $\mathcal{T}$ contains δ, and T^∞ is well-defined. We now claim that

$$T^\infty = \infty.$$

Proof of claim: Suppose to the contrary that $T^\infty < \infty$. Then from the continuity of $D(\Theta(\cdot))$,

$$\lim_{t \to T^\infty -} D(\Theta(t)) = D^\infty. \tag{3.8}$$

We now consider the temporal evolution of k_{ij}:

$$\dot{k}_{ij} = \mu \cos(\theta_j - \theta_i) - \gamma k_{ij} \geq \mu \cos D(\Theta) - \gamma k_{ij} \geq \mu \cos D^\infty - \gamma k_{ij}, \quad t \in [0, T^\infty).$$

We use the first condition (i) to obtain

$$\dot{k}_{ij} \geq \mu \cos D^\infty - \gamma k_{ij} = \gamma(k_m^0 - k_{ij}), \quad t \in [0, T^\infty).$$

This yields

$$k_{ij}(t) \geq k_m^0 + (k_{ij}^0 - k_m^0) e^{-\gamma t}, \quad t \in [0, T^\infty).$$

This again implies

$$k_{ij}(t) \geq k_m^0, \quad \text{i.e.,}\ k_m(t) \geq k_m^0, \quad t \in [0, T^\infty).$$

By the analyticity of phase difference $\theta_j - \theta_i$, the relative phase differences between oscillators can have finite number of zeros in a given finite time interval. We denote this finite number of zeros in phase differences by t_i in the time interval $[0, T^\infty]$:

$$0 = t_0 < t_1 < t_2 < \cdots < t_n = T^\infty.$$

In the time interval (t_{k-1}, t_k), we use

$$\sin(\theta_j - \theta_M) < R^\infty(\theta_j - \theta_M) < 0, \quad \text{where } R^\infty := \frac{\sin D^\infty}{D^\infty},\ -D^\infty \leq \theta_j - \theta_M \leq 0,$$

to obtain

$$\begin{aligned} \frac{d\theta_M}{dt} &= \omega_M + \sum_{j=1}^N k_{Mj} \sin(\theta_j - \theta_M) \\ &\leq \omega_M + k_m^0 R^\infty \sum_{j=1}^N (\theta_j - \theta_M) \\ &= \omega_M - N k_m^0 R^\infty \theta_M. \end{aligned}$$

Hence, we have

$$\frac{d\theta_M}{dt} \leq \omega_M - N k_m^0 R^\infty \theta_M, \quad \text{a.e. } [0, T^\infty). \tag{3.9}$$

Similarly, it follows that

$$\frac{d\theta_m}{dt} \geq \omega_m - N k_m^0 R^\infty \theta_m, \quad \text{a.e. } [0, T^\infty). \tag{3.10}$$

Thus, combining (3.9) and (3.10) yields

$$\frac{dD(\Theta)}{dt} \leq D(\Omega) - N k_m^0 R^\infty D(\Theta), \quad \text{a.e. } t \in [0, T^\infty).$$

By Gronwall's inequality,

$$\begin{aligned} D(\Theta(t)) &\leq D(\Theta^0) e^{-N k_m^0 R^\infty t} + \frac{D(\Omega)}{N k_m^0 R^\infty}\Big(1 - e^{-N k_m^0 R^\infty t}\Big) \\ &= \frac{D(\Omega)}{N k_m^0 R^\infty} + \Big(D(\Theta^0) - \frac{D(\Omega)}{N k_m^0 R^\infty}\Big) e^{-N k_m^0 R^\infty t} \end{aligned}$$

$$
\begin{aligned}
&< \frac{D(\Omega)}{Nk_m^0 R^\infty} + \Big(D(\Theta^0) - \frac{D(\Omega)}{Nk_m^0 R^\infty} \Big) \\
&= D(\Theta^0), \quad t \in [0, T^\infty),
\end{aligned}
$$

where we use the condition on K^0 in $(2.4)_2$ which implies

$$
k_m^0 > \frac{D(\Omega) D^\infty}{N D(\Theta^0) \sin D^\infty}.
$$

Thus, we have

$$
D(\Theta^0) - \frac{D(\Omega)}{Nk_m^0 R^\infty} > 0,
$$

which is contradictory to (3.8). Therefore, we have

$$
T^\infty = \infty.
$$

□

Based on Lemma 3.2, we are ready to provide our second result.

Proof of Theorem 2.2: It follows from Lemma 2.4 and Lemma 3.2 that

$$
\frac{d\|\Theta\|}{dt} \le \|\Omega\| - NR^\infty k_m \|\Theta\| \le \|\Omega\| - NR^\infty k_m^0 \|\Theta\|.
$$

This yields

$$
\|\Theta(t)\| \le \frac{\|\Omega\|}{NR^\infty k_m^0} + \Big(\|\Theta^0\| - \frac{\|\Omega\|^2}{NR^\infty k_m^0} \Big) e^{-R^\infty k_m^0 t}. \tag{3.11}
$$

Thus, we have

$$
\limsup_{t \to \infty} \|\Theta(t)\| \le \frac{\|\Omega\|}{Nk_m^0 R^\infty}.
$$

On the other hand, the relation $|\theta_i - \theta_j| \le \sqrt{2}\|\Theta\|$, i.e., $D(\Theta) \le \sqrt{2}\|\Theta\|$, implies

$$
\limsup_{t \to \infty} D(\Theta(t)) \le \frac{\sqrt{2}\|\Omega\|}{Nk_m^0 R^\infty}.
$$

This implies the desired result, which completes the proof.

In the next two sections, we study the second adaptive coupling model with $\Gamma_s(\theta) = \mu|\sin\theta|$, which is rather difficult to analyze compared to adaptive coupling Model A. Thus, in the following two sections, we consider two-body and many-body systems separately. The former can be analyzed with a detailed decay rate.

4. Synchronization of Model B: a two-body system

In this section, we first consider the adaptive coupling Model B with $N = 2$, which is analytically treatable unlike many-body systems where $N \ge 3$. and then

We consider a two-oscillator case with $\omega_2 > \omega_1$:

$$\begin{aligned}\dot\theta_1 &= \omega_1 + k_{12}\sin(\theta_2-\theta_1), \quad t>0,\\ \dot\theta_2 &= \omega_2 + k_{21}\sin(\theta_1-\theta_2),\\ \dot k_{12} &= \mu|\sin(\theta_2-\theta_1)| - \gamma k_{12}.\end{aligned} \tag{4.1}$$

To simplify system (4.1), we set

$$k := k_{12}, \quad \theta := \theta_2 - \theta_1, \quad \omega := \omega_2 - \omega_1 > 0.$$

In this case, system (4.1) becomes

$$\begin{aligned}\dot\theta &= \omega - 2k\sin\theta, \quad t>0,\\ \dot k &= \mu|\sin\theta| - \gamma k.\end{aligned} \tag{4.2}$$

Again, system (4.2) can be rewritten as an integro-differential equation:

$$\dot\theta = \omega - 2k^0 e^{-\gamma t}\sin\theta - 2\mu\Big[\int_0^t |\sin\theta(s)|e^{\gamma(s-t)}ds\Big]\sin\theta, \quad t>0. \tag{4.3}$$

Note that the differences $\theta = \theta_1 - \theta_2$, $\omega = \omega_1 - \omega_2$, and $k = k_{12}$ satisfy

$$\begin{aligned}\dot\theta &= \omega - 2k\sin\theta, \quad t>0,\\ \dot k &= \mu|\sin\theta| - \gamma k.\end{aligned} \tag{4.4}$$

4.1. **Identical oscillators.** In this subsection, we first consider the simple case when two oscillators are identical in the sense that

$$\omega_1 = \omega_2, \quad \text{i.e.,} \quad \omega = 0.$$

Thus, system (4.2) becomes

$$\dot\theta = -2k\sin\theta, \qquad \dot k = \mu|\sin\theta| - \gamma k, \qquad t>0. \tag{4.5}$$

It is easy to verify that the only equilibrium solution to (4.5) is $(0,0)$. Next, we show that this equilibrium is, in fact, asymptotically stable.

Proposition 4.1. (Asymptotic stability) *Let (θ, k) be a solution to system* (4.5) *with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then, we have

$$\lim_{t\to\infty}\theta(t) = 0, \qquad \lim_{t\to\infty} k(t) = 0.$$

Proof. First, note that the first equation of (4.5) yields

$$0 \le \theta(t) \le \theta^0 < \pi, \quad t \ge 0. \tag{4.6}$$

Thus, in this regime, $|\sin\theta| = \sin\theta$ and system (4.5) become

$$\dot\theta = -2k\sin\theta, \qquad \dot k = \mu\sin\theta - \gamma k, \quad t>0. \tag{4.7}$$

Note that the coupling strength always is nonnegative:

$$k(t) \ge 0, \quad t>0.$$

We first multiply the first equation of (4.7) by 2θ and use (4.6) and

$$\frac{\sin\theta^0}{\theta^0}\theta \le \sin\theta \le \theta, \quad t \ge 0,$$

to obtain

$$-2k\theta^2 \le \frac{d\theta^2}{dt} \le -2k\frac{\sin\theta^0}{\theta^0}\theta^2.$$

Gronwall's inequality implies

$$\theta^0 \exp\Big[-\int_0^t k(s)ds\Big] \le \theta(t) \le \theta^0 \exp\Big[-\frac{\sin\theta^0}{\theta^0}\int_0^t k(s)ds\Big]. \tag{4.8}$$

Note that (4.8) yields

$$\lim_{t\to\infty}\theta(t) = 0 \qquad \Longleftrightarrow \qquad \int_0^\infty k(s)ds = \infty. \tag{4.9}$$

On the other hand, since θ^2 is a nonincreasing function of t and bounded below by 0,

$$\exists\ \theta^* := \lim_{t\to\infty}\theta(t).$$

We now claim that

$$\theta^* = 0. \tag{4.10}$$

Proof of claim (4.10): Suppose to the contrary that $\theta^* \neq 0$. Then it follows from (4.9) that

$$\theta^* > 0, \quad \text{or equivalently,} \quad \int_0^\infty k(s)ds < \infty. \tag{4.11}$$

On the other hand, it follows from the dynamics of k that

$$\dot{k} = \mu\sin\theta - \gamma k \ge \mu\sin\theta^* - \gamma k.$$

Again, we have

$$k(t) \ge \Big(k^0 - \frac{\mu\sin\theta^*}{\gamma}\Big)e^{-\gamma t} + \frac{\mu\sin\theta^*}{\gamma}, \quad t \ge 0. \tag{4.12}$$

Integrating (4.12) from $t = 0$ to $t = \infty$ yields

$$\int_0^\infty k(s)ds = \infty,$$

which is contradictory to (4.11). Hence, we obtain the desired result. □

Remark 4.1. *The equilibrium* $(\theta, k) = (0, 0)$ *is not linearly asymptotically stable, which can be directly inferred. Consider a linearized system of* (4.5) *near* $(0, 0)$*:*

$$\dot{\theta} = 0, \quad \dot{k} = \mu\theta - \gamma k, \quad t > 0.$$

By direct calculations, we have

$$\theta(t) = \theta^0, \quad k(t) = \Big(k^0 - \frac{\mu\theta^0}{\gamma}\Big)e^{-\gamma t} + \frac{\mu\theta^0}{\gamma}, \quad t > 0.$$

Thus,

$$\lim_{t\to\infty}(\theta(t), k(t)) = \Big(\theta^0, \frac{\mu\theta^0}{\gamma}\Big).$$

Hence, synchronization is a nonlinear effect.

In Proposition 4.1 we studied the emergence of synchronization without detailed decay estimates. Next, we study the relaxation estimates of forward synchronization.

Proposition 4.2. (Short-time estimate) *Let (θ, k) be a solution to system (4.5) with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then for any $\alpha \in (0, k^0)$, there exists a positive time $t_* = t_*(\alpha)$ and $C_* = C_*(k^0, \theta^0, \alpha)$ such that

$$|\theta(t)| \le |\theta^0| e^{-\alpha R_0 t}, \qquad k(t) \le C_* e^{-\min\{\alpha R_0, \gamma\} t}, \quad t \le t_*(\alpha),$$

where $R_0 := \dfrac{\sin\theta^0}{\theta^0}$

Proof. (i) Note that Proposition 4.1 implies that there exists a $t_*(\alpha) > 0$ such that

$$k(t) \ge \alpha, \quad t \in [0, t_*(\alpha)).$$

Together, (4.5) and (4.6) imply

$$\dot\theta = -k\sin\theta \le -\alpha R_0 \theta, \quad t \le t_*(\alpha).$$

This yields the desired exponential decay estimate:

$$|\theta(t)| \le |\theta^0| e^{-\alpha R_0 t}, \quad t \le t_*(\alpha).$$

(ii) It follows from the equation for k in (4.5) that

$$\dot k = \mu|\sin\theta| - \gamma k \le \mu|\theta| - \gamma k \le \mu|\theta^0| e^{-\alpha R_0 t} - \gamma k, \quad t \le t_*(\alpha).$$

Gronwall's lemma yields the desired decay estimate for $k(t)$. □

Below, we study the large-time decay estimates of θ and k, which require the following lemma.

Lemma 4.1. *Let (θ, k) be a solution to system (4.5) with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then,

$$\lim_{t\to\infty} \frac{k(t)}{\theta(t)} = \frac{\mu}{\gamma}.$$

Proof. Consider the following system:

$$\dot\theta = -2k\sin\theta, \qquad \dot k = \mu|\sin\theta| - \gamma k, \qquad t > 0. \tag{4.13}$$

It follows from Proposition 4.1 that for $\varepsilon \ll 1$, there exists $t^* = t^*(\varepsilon) > 0$ such that

$$(1-\varepsilon)\theta \le \sin\theta \le (1+\varepsilon)\theta, \quad t \ge t^*. \tag{4.14}$$

Now, we consider the derivative of $\frac{k}{\theta}$:

$$\begin{aligned} \left(\frac{k}{\theta}\right)' &= \frac{\dot k\theta - k\dot\theta}{\theta^2} = \frac{(\mu\sin\theta - \gamma k)\theta - k(-2k\sin\theta)}{\theta^2} \\ &= \mu\frac{\theta\sin\theta}{\theta^2} - \gamma\frac{k}{\theta} + \frac{2k^2\sin\theta}{\theta^2}. \end{aligned} \tag{4.15}$$

• Case A (Lower bound): For $t \geq t^*$, we use the estimate $\sin\theta \geq (1-\varepsilon)\theta$ to obtain

$$\left(\frac{k}{\theta}\right)' \geq \mu(1-\varepsilon) - \gamma\frac{k}{\theta} + (1-\varepsilon)\frac{2k^2}{\theta}, \quad t \geq t^*.$$

Setting

$$Z := \frac{k}{\theta},$$

Z satisfies

$$Z' \geq \mu(1-\varepsilon) + \Big(2(1-\varepsilon)k - \gamma\Big)Z \geq \mu(1-\varepsilon) - \gamma Z, \quad t \geq t^*.$$

This yields

$$Z(t) \geq \frac{\mu(1-\varepsilon)}{\gamma} + \Big[Z(t^*) - \frac{\mu(1-\varepsilon)}{\gamma}\Big]e^{-\gamma(t-t^*)}, \quad t \geq t^*.$$

Letting $t \to \infty$, we obtain

$$\liminf_{t\to\infty} Z(t) \geq \frac{\mu(1-\varepsilon)}{\gamma}. \tag{4.16}$$

• Case B (Upper bound): We use (4.14) and (4.15) to derive

$$Z' \leq \mu(1+\varepsilon) + \Big(2(1+\varepsilon)k - \gamma\Big)Z, \quad t \geq t^*. \tag{4.17}$$

By Proposition 4.1, there exists $\tilde{t}^* > t^*$ such that

$$k < \frac{\varepsilon}{2(1+\varepsilon)}\gamma, \qquad t \geq \tilde{t}^*. \tag{4.18}$$

Combining (4.17) and (4.18) yields

$$Z' \leq \mu(1+\varepsilon) - (1-\varepsilon)\gamma Z, \quad t \geq \tilde{t}^*.$$

If follows from the same argument as the lower bound that

$$\limsup_{t\to\infty} Z(t) \leq \left(\frac{\mu}{\gamma}\right)\frac{1+\varepsilon}{1-\varepsilon}. \tag{4.19}$$

Letting $\varepsilon \to 0$ in (4.16) and (4.19) yields

$$\frac{\mu}{\gamma} \leq \liminf_{t\to\infty} Z(t) \leq \limsup_{t\to\infty} Z(t) \leq \frac{\mu}{\gamma},$$

which gives the desired result. $\square$

Proposition 4.3. *Let (θ, k) be a solution to system (4.5) with initial data satisfying*

$$0 < \theta^0 < \pi, \quad k^0 > 0.$$

Then there exists a sufficiently large $\bar{t}^*$ such that θ and k decay to zero like $\frac{1}{t+1}$ for $t > \bar{t}^*$.

Proof. • Case A (Temporal-decay estimate of θ): By Lemma 4.1, for any small $\varepsilon \ll 1$, there exists $\hat{t}^* \gg 1$ such that

$$(1-\varepsilon)\frac{\mu}{\gamma} \leq \frac{k}{\theta} \leq (1+\varepsilon)\frac{\mu}{\gamma}, \quad (1-\varepsilon)\theta \leq \sin\theta \leq (1+\varepsilon)\theta, \quad t \geq \hat{t}^*. \tag{4.20}$$

The first equations of (4.13) and (4.20) imply

$$-\frac{2(1+\varepsilon)^2\mu}{\gamma}\theta^2 \leq \dot{\theta} \leq -\frac{2(1-\varepsilon)^2\mu}{\gamma}\theta^2, \quad t \geq \hat{t}^*.$$

This yields

$$\frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1+\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)} \le \theta(t) \le \frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1-\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)}, \qquad t \ge \hat{t}^*.$$

• Case B (Temporal-decay estimate of k): We use (4.20) to see that

$$k(t) \le (1+\varepsilon)\frac{\mu}{\gamma}\theta \le (1+\varepsilon)\frac{\mu}{\gamma}\frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1-\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)}, \quad t \ge \hat{t}^*. \tag{4.21}$$

On the other hand,

$$k(t) \ge (1-\varepsilon)\frac{\mu}{\gamma}\theta \ge (1-\varepsilon)\frac{\mu}{\gamma}\frac{\gamma\theta(\hat{t}^*)}{\gamma + 2(1+\varepsilon)^2\mu\theta(\hat{t}^*)(t-\hat{t}^*)}, \quad t \ge \hat{t}^*. \tag{4.22}$$

Finally, by combining (4.21) and (4.22), we derive the desired result. □

Remark 4.2. *It follows from Propositions 4.2 and 4.3 that θ and k exhibit two stages in the relaxation process: a fast exponential relaxation stage for small-time and a slow algebraic relaxation stage for large-time. For a constant coupling strength $k(t) = k^\infty$ and generic initial data, it is well-known that relaxation toward complete synchronization is always exponential* [12]. *Thus, competition between θ and k makes the relaxation asymptotically slow.*

4.2. **Nonidentical oscillators.** In this subsection, we study complete synchronization of a two-body system with non-identical natural frequencies $\omega_1 \neq \omega_2$.

We first look for equilibria $(\theta^\infty, k^\infty)$ of (4.4):

$$\omega - 2k^\infty \sin\theta^\infty = 0, \quad \mu|\sin\theta^\infty| - \gamma k^\infty = 0, \quad \omega > 0.$$

This yields

$$k^\infty = \frac{\mu}{\gamma}|\sin\theta^\infty|, \qquad |\sin\theta^\infty|\sin\theta^\infty = \frac{\gamma\omega}{2\mu}. \tag{4.23}$$

Note that system (4.23) is solvable if and only if

$$\theta^\infty \in [0,\pi], \qquad \omega \le \frac{2\mu}{\gamma}.$$

In this case, equilibria are determined as follows.

If $\omega < \frac{2\mu}{\gamma}$, then there are two equilbria:

$$(\theta^{1\infty}, k^{1\infty}) = \Big(\arcsin\sqrt{\frac{\gamma\omega}{2\mu}}, \sqrt{\frac{\mu\omega}{2\gamma}}\Big), \quad (\theta^{2\infty}, k^{2\infty}) = \Big(\pi - \arcsin\sqrt{\frac{\gamma\omega}{2\mu}}, \sqrt{\frac{\mu\omega}{2\gamma}}\Big).$$

If $\omega = \frac{2\mu}{\gamma}$,

$$(\theta^{3\infty}, k^{3\infty}) = \Big(\frac{\pi}{2}, \frac{\mu}{\gamma}\Big).$$

We now perform a linear stability analysis for system (4.2). First, we set

$$\bar{\theta} := \theta - \theta^\infty, \quad \bar{k} := k - k^\infty.$$

Then the linearized system for $(\bar{\theta}, \bar{k})$ is given by

$$\frac{d}{dt}\begin{pmatrix} \bar{\theta} \\ \bar{k} \end{pmatrix} = \begin{pmatrix} -k^\infty\cos\theta^\infty & -\sin\theta^\infty \\ \mu\cos\theta^\infty & -\gamma \end{pmatrix}\begin{pmatrix} \bar{\theta} \\ \bar{k} \end{pmatrix}. \tag{4.24}$$

By direct calculations, the characteristic polynomial of the coefficient matrix of (4.24) is

$$p_1(\lambda) = \lambda^2 + (\gamma + k^\infty \cos\theta^\infty)\lambda + \gamma k^\infty \cos\theta^\infty + \mu \sin\theta^\infty \cos\theta^\infty. \tag{4.25}$$

Now, let λ_1 and λ_2 be the roots of (4.25), i.e., the eigenvalues of the coefficient matrix in (4.24). To determine the sign of the real parts of λ_i, we consider two cases.

- Case A (supercritical case, $\omega < \frac{2\mu}{\gamma}$): For $(\theta^{1\infty}, k^{1\infty})$, since $\cos\theta^{1\infty} > 0$, it follows that

$$\begin{aligned} &\lambda_1 + \lambda_2 = -(\gamma + k^\infty \cos\theta^\infty) < 0, \\ &\lambda_1\lambda_2 = \gamma k^\infty \cos\theta^\infty + \mu \sin\theta^\infty \cos\theta^\infty > 0. \end{aligned} \tag{4.26}$$

Since the polynomial $p_1(\lambda)$ has real coefficients, it has two real roots or two conjugate complex roots. Thus, (4.26) implies that the real parts of the two eigenvalues are negative in both cases. Hence, $(\theta^{1\infty}, k^{1\infty})$ is a linearly stable hyperbolic equilibrium.

On the other hand, for $(\theta^{2\infty}, k^{2\infty})$, since $\cos\theta^{2\infty} < 0$, it follows that

$$\lambda_1\lambda_2 = \gamma k^\infty \cos\theta^\infty + \mu \sin\theta^\infty \cos\theta^\infty = \cos\theta^\infty(\gamma k^\infty + \mu \sin\theta^\infty) < 0.$$

This implies that the polynomial $p_1(\lambda)$ does not have two conjugate complex roots, and the linearized system (4.24) has one positive eigenvalue and one negative eigenvalue. Hence, $(\bar\theta_2, \bar k)$ is linearly unstable.

- Case B (critical case, $\omega = \frac{2\mu}{\gamma}$) : In this case, only one equilibrium exists:

$$(\theta^{3\infty}, k^{3\infty}) = \Big(\frac{\pi}{2}, \frac{\mu}{\gamma}\Big).$$

The corresponding linearized system at $(\theta^{3\infty}, k^{3\infty})$ is as follows.

$$\dot{\bar\theta} = -\bar k, \qquad \dot{\bar k} = -\gamma \bar k, \quad t > 0.$$

Thus, by direct calculation, we have

$$\bar k(t) = \bar k^0 e^{-\gamma t}, \qquad \bar\theta(t) = \bar\theta^0 - \frac{\bar k^0}{\gamma} + \frac{\bar k^0}{\gamma} e^{-\gamma t}, \quad t \geq 0.$$

Note that

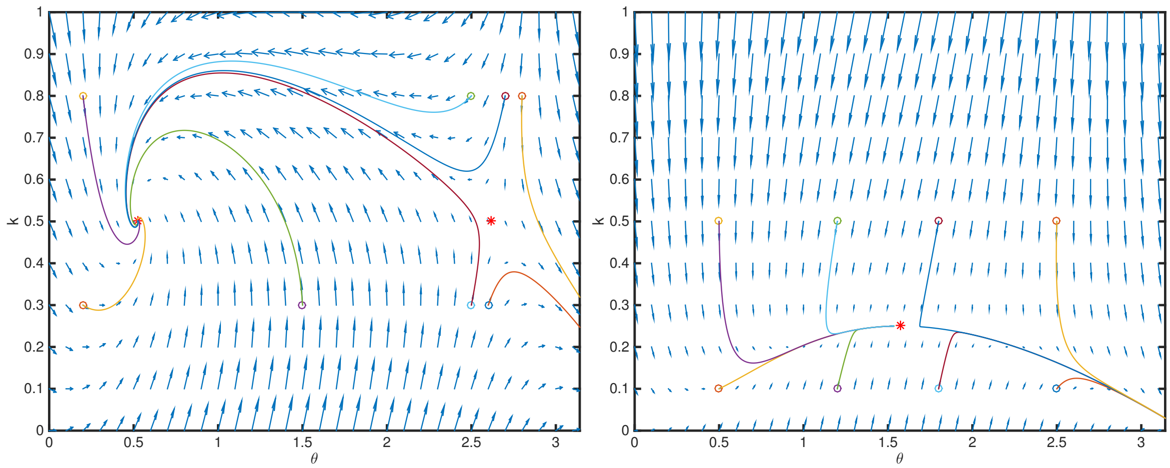
$$\lim_{t\to\infty} (\bar k(t), \bar\theta(t)) = \Big(0, \bar\theta^0 - \frac{\bar k^0}{\gamma}\Big).$$

Thus the unique equilibrium $(\theta^{3\infty}, k^{3\infty})$ is not asymptotically stable. This can be seen easily from two numerical simulations to assess the system's behavior. The results are shown in Fig. 3.

5. Synchronization estimate of Model B: a many-body system

In this section, we study the complete synchronization problem for the adaptive coupling Model B with $N \geq 3$. As in the previous section, we provide a rigorous result for complete synchronization of identical oscillators; in contrast, for nonidentical oscillators, we provide complete synchronization under he a priori assumption:

$$\sup_{0\leq t<\infty} D(\Theta(t)) < \frac{\pi}{2}.$$



(a) Phase portrait for the supercritical case (b) Phase portrait for the critical case

FIGURE 3. (a) $(\mu, \gamma, \omega) = (1, 1, 0.5)$, $(b)(\mu, \gamma, \omega) = (1, 4, 0.5)$.

5.1. **Identical oscillators.** In this subsection, we present a synchronization estimate for (2.5). We assume that

$$\omega_i = 0, \quad 1 \le i \le N.$$

In this case, system (2.5) becomes

$$\begin{aligned} \dot{\theta}_i &= \sum_{j=1}^{N} k_{ij} \sin(\theta_j - \theta_i), \quad t > 0, \quad 1 \le i \le N, \\ \dot{k}_{ij} &= \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}. \end{aligned} \tag{5.1}$$

We set extremal indices M and m as follows:

$$\theta_M := \max_{1 \le i \le N} \theta_i, \qquad \theta_m := \min_{1 \le i \le N} \theta_i.$$

For a solution (Θ, K) to (5.1), we set three Lyapunov functionals:

$$\begin{aligned} \mathcal{L}_1(t) &:= D(\Theta(t)) + \frac{1}{2\mu} \sum_{j=1}^{N} (k_{Mj}^2 + k_{mj}^2), \quad t \ge 0, \\ \mathcal{L}_2(t) &:= \sum_{i=1}^{N} |\theta_i|^2 + \frac{2\gamma}{3\mu^2} \sum_{i,j=1}^{N} k_{ij}^3, \\ \mathcal{L}_3(t) &:= \sum_{i=1}^{N} |\dot{\theta}_i|^2 + \frac{\mu}{4} \sum_{i,j=1}^{N} \mathrm{sgn}(\theta_i - \theta_j) \Big(2(\theta_i - \theta_j) - \sin 2(\theta_i - \theta_j) \Big). \end{aligned} \tag{5.2}$$

In the following lemma, we study the time-variation of the functionals in (5.2).

Lemma 5.1. *Let* (Θ, K) *be a solution to* (5.1) *with initial data satisfying* $D(\Theta^0) < \frac{\pi}{2}$*. Then the functionals in* (5.2) *are nonincreasing along the flow* (5.1)*:*

$$(i)\ \mathcal{L}_1(t) + \frac{\gamma}{\mu}\sum_{j=1}^{N}\int_0^t \Big(k_{Mj}^2(s) + k_{mj}^2(s)\Big) ds = \mathcal{L}_1(0), \quad t > 0.$$

$$(ii)\ \mathcal{L}_2(t) + \frac{1}{\mu^2}\sum_{i,j=1}^{N}\int_0^t k_{ij}(s)\Big(|\dot{k}_{ij}(s)|^2 + \gamma k_{ij}^2(s)\Big) ds \le \mathcal{L}_2(0).$$

$$(iii)\ \mathcal{L}_3(t) + 2\int_0^t \|\dot{\Theta}(s)\|_2^2 ds + \sum_{i,j=1}^{N}\int_0^t k_{ij}(s)\cos(\theta_i - \theta_j)|\dot{\theta}_i - \dot{\theta}_j|^2 ds = \mathcal{L}_3(0).$$

Proof. Suppose $D(\Theta^0) < \frac{\pi}{2}$. It follows from Lemma 2.3 that

$$D(\Theta(t)) \le D(\Theta^0) < \frac{\pi}{2}, \quad t \ge 0.$$

(i) It follows from (5.1) that

$$\frac{d}{dt}D(\Theta(t)) = \dot{\theta}_M - \dot{\theta}_m = \sum_{j=1}^{N} k_{Mj}\sin(\theta_j - \theta_M) - \sum_{j=1}^{N} k_{mj}\sin(\theta_j - \theta_m). \tag{5.3}$$

Since the phase difference $D(\Theta(t)) < \frac{\pi}{2}$, we obtain

$$\begin{aligned} \frac{d}{dt}\sum_{j=1}^{N}\frac{k_{Mj}^2}{2} &= \sum_{j=1}^{N} k_{Mj}\dot{k}_{Mj} = \sum_{j=1}^{N} k_{Mj}\Big(\mu|\sin(\theta_M - \theta_j)| - \gamma k_{Mj}\Big) \\ &= -\mu\sum_{j=1}^{N} k_{Mj}\sin(\theta_j - \theta_M) - \gamma\sum_{j=1}^{N} k_{Mj}^2. \end{aligned} \tag{5.4}$$

Similarly, we have

$$\begin{aligned} \frac{d}{dt}\sum_{j=1}^{N}\frac{k_{mj}^2}{2} &= \sum_{j=1}^{N} k_{mj}\dot{k}_{mj} = \sum_{j=1}^{N} k_{mj}(\mu|\sin(\theta_m - \theta_j)| - \gamma k_{mj}) \\ &= \mu\sum_{j=1}^{N} k_{mj}\sin(\theta_j - \theta_m) - \gamma\sum_{j=1}^{N} k_{mj}^2. \end{aligned} \tag{5.5}$$

The linear combination (5.3) + $\frac{1}{\mu} \times \Big((5.4) + (5.5)\Big)$ yields

$$\frac{d}{dt}\Big[D(\Theta(t)) + \frac{1}{2\mu}\sum_{j=1}^{N}(k_{Mj}^2 + k_{mj}^2)\Big] = -\frac{\gamma}{\mu}\sum_{j=1}^{N}(k_{Mj}^2 + k_{mj}^2).$$

This gives the desired result.

(ii) Multiplying (5.1) by $2\theta_i$, summing the result, and using the $i \leftrightarrow j$ exchange technique, we obtain

$$\frac{d}{dt}\sum_{i=1}^{N}\theta_i^2 = \sum_{i,j=1}^{N} k_{ij}\sin(\theta_j - \theta_i)(\theta_i - \theta_j).$$

Using $x\sin x \ge \sin^2 x$ for $x \in [-\pi, \pi]$, gives rise to the estimate

$$
\begin{aligned}
\frac{d}{dt}\sum_{i=1}^N \theta_i^2 &= -\sum_{i,j=1}^N k_{ij}\sin(\theta_j-\theta_i)(\theta_j-\theta_i) \\
&\le -\sum_{i,j=1}^N k_{ij}\sin^2(\theta_j-\theta_i) = -\sum_{i,j=1}^N \frac{k_{ij}}{\mu^2}(\dot{k}_{ij}+\gamma k_{ij})^2 \\
&= -\frac{1}{\mu^2}\sum_{i,j=1}^N k_{ij}|\dot{k}_{ij}|^2 - \frac{2\gamma}{\mu^2}\sum_{i,j=1}^N k_{ij}^2\dot{k}_{ij} - \frac{\gamma^2}{\mu^2}\sum_{i,j=1}^N k_{ij}^3 \\
&= -\frac{1}{\mu^2}\sum_{i,j=1}^N k_{ij}(|\dot{k}_{ij}|^2+\gamma^2 k_{ij}^2) - \frac{d}{dt}\sum_{i,j=1}^N \frac{2\gamma}{3\mu^2}k_{ij}^3.
\end{aligned} \tag{5.6}
$$

Integrating (5.6) yields the desired result.

(iii) We differentiate (5.1) to consider the following system for frequency $\dot{\theta}_i$:

$$
\begin{aligned}
\frac{d}{dt}\dot{\theta}_i &= \sum_{j=1}^N \dot{k}_{ij}\sin(\theta_j-\theta_i) + \sum_{j=1}^N k_{ij}\cos(\theta_j-\theta_i)(\dot{\theta}_j-\dot{\theta}_i) \\
&= \mu\sum_{j=1}^N |\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j) - \gamma\dot{\theta}_i - \sum_{j=1}^N k_{ij}\cos(\theta_i-\theta_j)(\dot{\theta}_i-\dot{\theta}_j).
\end{aligned} \tag{5.7}
$$

Multiplying (5.7) by $2\dot{\theta}_i$, summing the result, and using the $i \leftrightarrow j$ exchange technique yields

$$
\begin{aligned}
\frac{d}{dt}\sum_{i=1}^N \dot{\theta}_i^2 &= \mu\sum_{i,j=1}^N |\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j)(\dot{\theta}_i-\dot{\theta}_j) \\
&\quad - 2\gamma\sum_{i=1}^N \dot{\theta}_i^2 - \sum_{i,j=1}^N k_{ij}\cos(\theta_i-\theta_j)|\dot{\theta}_i-\dot{\theta}_j|^2 \\
&= -\frac{\mu}{4}\frac{d}{dt}\sum_{i,j=1}^N \Big\{\mathrm{sgn}(\theta_i-\theta_j)\Big(2(\theta_i-\theta_j)-\sin 2(\theta_i-\theta_j)\Big)\Big\} \\
&\quad - 2\gamma\sum_{i=1}^N \dot{\theta}_i^2 - \sum_{i,j=1}^N k_{ij}\cos(\theta_i-\theta_j)|\dot{\theta}_i-\dot{\theta}_j|^2,
\end{aligned} \tag{5.8}
$$

where the following relation is used:

$$
\begin{aligned}
&\frac{d}{dt}\Big\{\mathrm{sgn}(\theta_i-\theta_j)\Big(2(\theta_i-\theta_j)-\sin 2(\theta_i-\theta_j)\Big)\Big\} \\
&\qquad = \mathrm{sgn}(\theta_i-\theta_j)\Big(2(\dot{\theta}_i-\dot{\theta}_j)-2(\dot{\theta}_i-\dot{\theta}_j)(\cos 2(\theta_i-\theta_j))\Big) \\
&\qquad = 4\mathrm{sgn}(\theta_i-\theta_j)\Big((\dot{\theta}_i-\dot{\theta}_j)\sin^2(\theta_i-\theta_j)\Big) \\
&\qquad = 4(\dot{\theta}_i-\dot{\theta}_j)|\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j).
\end{aligned} \tag{5.9}
$$

Integrating (5.8) yields the desired result. □

Proof of Theorem 2.3: Let $\Theta = \Theta(t)$ be a solution to (5.1) with initial data Θ^0 satisfying $D(\Theta^0) < \frac{\pi}{2}$. Then it follows from Lemma 2.3 that

$$\sup_{t \ge 0} D(\Theta(t)) \le D(\Theta^0) < \pi.$$

• Case A: We first provide the estimates:

$$\lim_{t\to\infty} |k_{ij}(t)| = 0, \quad \lim_{t\to\infty} |\dot{\theta}_i(t)| = 0, \quad 1 \le i, j \le N.$$

⋄ Case A.1: For the zero convergence of the coupling strength, we use Lemma 5.1 to obtain

$$\int_0^\infty k_{ij}^3(s) ds \le \mathcal{L}_2(0) < \infty. \tag{5.10}$$

On the other hand, note that

$$\dot{k}_{ij} = \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij} \le \mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}.$$

This yields

$$k_{ij}(t) \le k_{ij}(0) e^{-\gamma t} + \frac{\mu}{\gamma}(1 - e^{-\gamma t}) \le k_{ij}(0) + \frac{\mu}{\gamma}.$$

It follows from (5.1) that

$$|\dot{k}_{ij}| = |\mu |\sin(\theta_j - \theta_i)| - \gamma k_{ij}| \le 2\mu + \gamma k_{ij}(0). \tag{5.11}$$

Then the estimate (5.11) and uniform boundedness of k_{ij} (see (ii) of Lemma 5.1) imply the uniform boundedness of $\frac{d}{dt} k_{ij}^3$:

$$\sup_{t \ge 0} \left| \frac{d}{dt} k_{ij}^3 \right| < \infty. \tag{5.12}$$

Finally, we combine (5.10) and (5.12) to conclude

$$\lim_{t\to\infty} k_{ij}(t) = 0, \quad \text{or equivalently,} \ \lim_{t\to\infty} |k_{ij}(t)| = 0. \tag{5.13}$$

This clearly yields

$$\sup_{t \ge 0} \max_{1 \le i,j \le N} k_{ij}(t) \le k^\infty.$$

⋄ Case A.2: It follows from Lemma 5.1 that

$$\begin{aligned} &\|\dot{\Theta}(t)\|_2^2 \le \mathcal{L}_3(t) \le \mathcal{L}_3(0) \qquad \text{and} \\ &2\int_0^t \|\dot{\Theta}(s)\|_2^2 ds + \sum_{i,j=1}^N \int_0^t k_{ij}(s) \cos(\theta_i - \theta_j) |\dot{\theta}_i - \dot{\theta}_j|^2 ds \le \mathcal{L}_3(0). \end{aligned} \tag{5.14}$$

Recall from (5.8) that

$$\begin{aligned}\frac{d}{dt}\|\dot\Theta\|_2^2 &= \mu\sum_{i,j=1}^N |\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j)(\dot\theta_i-\dot\theta_j)\\ &\quad -2\gamma\sum_{i=1}^N \dot\theta_i^2 - \sum_{i,j=1}^N k_{ij}\cos(\theta_i-\theta_j)|\dot\theta_i-\dot\theta_j|^2\\ &\le \mu\Big|\sum_{i,j=1}^N |\sin(\theta_i-\theta_j)|\sin(\theta_i-\theta_j)(\dot\theta_i-\dot\theta_j)\Big|\\ &\le \mu\sum_{i,j=1}^N (1+|\dot\theta_i-\dot\theta_j|^2) \le \mu\Big(N^2+2\|\dot\Theta\|_2^2\Big)\\ &\le \mu\Big(N^2+2\mathcal{L}_3(0)\Big).\end{aligned} \tag{5.15}$$

Finally, we combine (5.14) and (5.15) to obtain

$$\lim_{t\to\infty}\|\dot\Theta(t)\|_2 = 0.$$

• Case B: We first provide the estimate:

$$\lim_{t\to\infty} D(\Theta(t)) = 0.$$

Note that for all $1\le i,j\le N$,

$$\theta_i(t) - \ \theta_j(t) \text{ is uniformly continuous for } t\ge 0.$$

Since $D(\dot\Theta) = \dot\theta_M - \dot\theta_m \le 0$, we know that $D(\Theta)$ is nonincreasing and bounded below. Thus, there exists $D^*\ge 0$ such that

$$D^* := \lim_{t\to\infty} D(\Theta(t)).$$

Suppose that D^* is strictly positive, i.e.,

$$\lim_{t\to\infty}(\theta_M-\theta_m)\ge D^* > 0.$$

For each $t>0$, there is an (i,j) pair such that $|\theta_i(t)-\theta_j(t)|\ge D_*$. On the other hand, because of the uniform continuity of $\theta_i-\theta_j$, there is a positive constant $\delta>0$ such that $|\theta_i(s)-\theta_j(s)| > \frac{D^*}{2}$ for $s\in(t-\delta,t)$. We estimate that

$$\begin{aligned}k_{ij}(t) &= k_{ij}(0)e^{-\gamma t} + \mu\int_0^t |\sin(\theta_i(s)-\theta_j(s))|e^{-\gamma(t-s)}ds\\ &\ge \mu\int_{t-\delta}^t |\sin(\theta_i(s)-\theta_j(s))|e^{-\gamma(t-s)}ds\\ &\ge \mu\delta\sin\Big(\frac{D^*}{2}\Big)e^{-\gamma\delta},\end{aligned}$$

which contradicts (5.13). Therefore, we conclude that $D^*=0$. This completes the proof.

5.2. **Nonidentical oscillators.** We now return to the nonidentical case when $D(\Omega) > 0$. We introduce a fourth Lyapunov functional $\mathcal{L}_4$ as follows. For a global solution (Θ, K) to (5.1), we define

$$\mathcal{L}_4(t) := \sum_{i=1}^{N} \Big(|\dot\theta_i|^2 - 2\gamma\omega_i\theta_i \Big) + \frac{\mu}{4} \sum_{i,j=1}^{N} \mathrm{sgn}(\theta_i - \theta_j)\Big(2(\theta_i - \theta_j) - \sin 2(\theta_i - \theta_j)\Big). \tag{5.16}$$

Lemma 5.2. *For Let $\Theta = \Theta(t)$ be a global solution to (5.1) with the a priori assumption:*

$$\sup_{t\ge 0} D(\Theta(t)) < \frac{\pi}{2}.$$

Then, the functional $\mathcal{L}_4(t)$ satisfies

$$\mathcal{L}_4(t) + 2\int_0^t \|\dot\Theta\|^2 ds + \sum_{i,j=1}^{N} \int_0^t k_{ij} \cos(\theta_i - \theta_j)|\theta_i - \theta_j|^2 ds = \mathcal{L}_4(0).$$

Proof. We differentiate the first equation of (2.5) to consider the following:

$$\begin{aligned} \frac{d}{dt}\dot\theta_i &= \sum_{j=1}^{N} \dot k_{ij} \sin(\theta_j - \theta_i) + \sum_{j=1}^{N} k_{ij} \cos(\theta_j - \theta_i)(\dot\theta_j - \dot\theta_i) \\ &= \mu \sum_{j=1}^{N} |\sin(\theta_i - \theta_j)| \sin(\theta_i - \theta_j) - \gamma(\dot\theta_i - \omega_i) + \sum_{j=1}^{N} k_{ij} \cos(\theta_i - \theta_j)(\dot\theta_i - \dot\theta_j). \end{aligned} \tag{5.17}$$

Multiplying (5.17) by $2\dot\theta_i$, summing the result, and using the $i \leftrightarrow j$ exchange technique, we obtain

$$\begin{aligned} \frac{d}{dt}\sum_{i=1}^{N} \dot\theta_i^2 &= \mu \sum_{i,j=1}^{N} |\sin(\theta_i - \theta_j)| \sin(\theta_i - \theta_j)(\dot\theta_i - \dot\theta_j) \\ &\quad - 2\gamma\Big(\sum_{i=1}^{N} \dot\theta_i^2 - \sum_{i=1}^{N} \omega_i\dot\theta_i \Big) - \sum_{j=1}^{N} k_{ij} \cos(\theta_i - \theta_j)|\dot\theta_i - \dot\theta_j|^2 \\ &= -\frac{\mu}{4}\frac{d}{dt} \sum_{i,j=1}^{N} \Big\{ \mathrm{sgn}(\theta_i - \theta_j)\Big(2(\theta_i - \theta_j) - \sin 2(\theta_i - \theta_j)\Big) \Big\} \\ &\quad - 2\gamma \sum_{i=1}^{N} \dot\theta_i^2 - 2\gamma\frac{d}{dt} \sum_{i=1}^{N} \omega_i\theta_i - \sum_{j=1}^{N} k_{ij} \cos(\theta_i - \theta_j)|\dot\theta_i - \dot\theta_j|^2, \end{aligned} \tag{5.18}$$

where (5.9) is utilized. Integrating (5.18) yields the desired result. □

Proof of Theorem 2.4: Let $\Theta = \Theta(t)$ be a solution to (5.1) with the a priori assumption $D(\Theta(t)) < \frac{\pi}{2}$. We claim:

$$\lim_{t\to\infty} |\dot\theta_i(t)| = 0, \quad 1 \le i, j \le N.$$

With the assumption that $D(\Theta(t)) < \frac{\pi}{2}$ for $t \geq 0$, functional $\mathcal{L}_4(t)$ is positive for all t. By Lemma 5.2, we have

$$2\int_0^t \|\dot{\Theta}\|^2 ds + \int_0^t \sum_{i,j=1}^N \cos(\theta_i - \theta_j)|\theta_i - \theta_j|^2 ds = \mathcal{L}_4(0) - \mathcal{L}_4(t) \leq |\mathcal{L}_4(0)| + |\mathcal{L}_4(t)|. \tag{5.19}$$

We now show the uniform boundedness of $\mathcal{L}_4(t)$ and $\|\dot{\Theta}\|$. First, we recall the uniform boundedness of coupling strengths k_{ij}:

$$k_{ij}(t) \leq \frac{\mu}{\gamma} + \Big(k_{ij}(0) - \frac{\mu}{\gamma}\Big)e^{-\gamma t} \leq k^\infty.$$

Since $D(\Theta(t)) < \dfrac{\pi}{2}$ and $\displaystyle\sum_{i=1}^N \theta_i = 0$, we have

$$\sum_{i=1}^N |\theta_i| \leq \frac{N\pi}{2}.$$

Thus, we attain

$$\begin{aligned} |\mathcal{L}_4(t)| &\leq \sum_{i=1}^N |\dot{\theta}_i|^2 + 2\gamma\|\Omega\|_\infty \sum_{i=1}^N (|\frac{\pi}{2}| + |\theta_i|) + \frac{\mu}{4}\sum_{i,j=1}^N (2D(\Theta(t)) + 1) \\ &\leq N(\|\Omega\|_\infty + K^\infty N)^2 + 2\gamma\|\Omega\|_\infty N\pi + \frac{\mu N^2(\pi+1)}{4}. \end{aligned}$$

Hence,

$$\sup_{t\geq 0}|\mathcal{L}_4(t)| \leq N(\|\Omega\|_\infty + K^\infty N)^2 + 2\gamma\|\Omega\|_\infty N\pi + \frac{\mu N^2(\pi+1)}{4} := C_1^\infty.$$

This implies

$$\|\dot{\Theta}\|^2 \leq \mathcal{L}^4(t) + \sum_{i=1}^N 2\gamma\omega_i\theta_i \leq C_1^\infty + 2\gamma\|\Omega\|\|\theta\| := C_2^\infty.$$

It suffices to show that $\frac{d}{dt}\|\dot{\Theta}(t)\|^2$ is uniformly bounded:

$$\begin{aligned} \frac{d}{dt}\|\dot{\Theta}\|^2 &= \mu\sum_{i,j=1}^N |\sin(\theta_i - \theta_j)|\sin(\theta_i - \theta_j)(\dot{\theta}_i - \dot{\theta}_j) \\ &\quad - 2\gamma\sum_{i=1}^N \dot{\theta}_i^2 - 2\gamma\sum_{i=1}^N \omega_i\dot{\theta}_i - \sum_{i,j=1}^N k_{ij}\cos(\theta_i - \theta_j)|\dot{\theta}_i - \dot{\theta}_j|^2 \\ &\leq \mu\Big|\sum_{i,j=1}^N |\sin(\theta_i - \theta_j)|\sin(\theta_i - \theta_j)(\dot{\theta}_i - \dot{\theta}_j)\Big| + 2\gamma\sum_{i=1}^N |\omega_i||\dot{\theta}_i| \\ &\leq \mu\Big(N^2 + 2\|\dot{\Theta}\|_2^2\Big) + 2\gamma\|\Omega\|\|\dot{\Theta}\| \\ &\leq \mu\Big(N^2 + 2C_2^\infty\Big) + 2\gamma\|\Omega\|\sqrt{C_2^\infty}. \end{aligned} \tag{5.20}$$

Therefore, from (5.19) and (5.20), we conclude that

$$\lim_{t\to\infty} \|\dot{\Theta}\|_2 = 0.$$

This completes the proof of Theorem 2.4.

6. Conclusions

In this paper, we presented two generalized Kuramoto models with adaptive couplings in coupling strength. The classical Kuramoto model governs the dynamics of phases of weakly coupled oscillators with constant all-to-all coupling strength. However, in many realistic applications arising from neuroscience, the assumption of constant and uniform coupling strengths is too restrictive. In fact, the pairwise coupling strength is adaptive to the degree of phase synchronization. Thus, several adaptive coupling models were proposed and investigated numerically in physics and control theory literature. In this paper, we considered two distinct coupling laws describing the coupling, and presented sufficient frameworks leading to weak and strong synchronizations. For the cosine adaptive coupling law consistent with Hebbian learning law, we showed that for identical oscillators, as long as initial phases are confined inside an arc with geodesic length strictly less than $\frac{\pi}{2}$, the phase diameter decays to zero at least exponentially fast, i.e., ACS holds (Theorem 2.1), whereas for nonidentical oscillators, the strong synchronization (ACS) does not hold, but the phase-diameter is the inverse order of initial minimum coupling strength in a large time regime so that APS occurs (Theorem 2.2). On the other hand, for sinusoidal adaptive coupling law, as phase synchronization is achieved, the coupling strengths also tend to zero asymptotically. Thus, the analysis is more delicate compared to cosine adaptive law. This can be easily seen in the simple system consisting of two oscillators with the same natural frequencies. In a small-time regime, the relative phase difference and coupling strength decay exponentially (Proposition 4.2) and then in a large-time regime, the relative phase difference and coupling strength decay to zero at the order of $(1+t)^{-1}$. Thus, the multi-scale phenomenon "transition from fast relaxation to slow relaxation" is observed (Proposition 4.3). However, for a many-body system with $N \geq 3$, we cannot extend two-oscillator results to many-oscillator system directly. For identical oscillators, we instead employed a Lyapunov functional approach. Through the time-evolution estimates of functionals, we showed that for an initial phase configurations confined on the half-circle, asymptotic complete synchronization occurs and the pairwise coupling strengths tend to zero simultaneously (Theorem 2.3). On the other hand, for nonidentical oscillators, we can only obtain a conditional result, namely, as long as the configuration is confined inside some arc with length less than $\frac{\pi}{2}$, asymptotic complete synchronization holds (Theorem 2.4). Thus, finding conditions for initial phase configurations and parameters to guarantee a priori bounding condition remains an open problem. More over, in this paper we only considered two specific adaptive couplings. Thus, extending our results to more general adaptive systems is left for a future work,

References

[1] J. A. Acebron, L. L. Bonilla, C. J. P. Pérez Vicente, F. Ritort and R. Spigler, *The Kuramoto model: A simple paradigm for synchronization phenomena,* Rev. Mod. Phys., 77 (2005), pp. 137–185.

[2] T. Aoki and T. Aoyagi, *Coevolution of phases and connection strengths in a network of phase oscillators,* Phys. Rev. Lett., 102 (2009), pp. 034101.

[3] J. C. Bronski, L. DeVille and M. J. Park, *Fully Synchronous Solutions and the Synchronization Phase Transition for the Finite N Kuramoto Model,* Chaos, 22 (2012), pp. 033133.

[4] Y.-P. Choi, S.-Y. Ha, S. Jung and Y. Kim, *Asymptotic formation and orbital stability of phase-locked states for the Kuramoto model,* Physica D, 241 (2012), pp. 735–754.
[5] Y.-P. Choi, S.-Y. Ha, M.-M. Kang and M. Kang, *Exponential synchronization of finite-dimensional Kuramoto model at critical coupling strength,* Commun. Math. Sci., 11 (2013), pp. 385–401.
[6] D. Cumin and C. P. Unsworth, *Generalizing the Kuramoto model for the study of neuronal synchronization in the brain,* Physica D, 226 (2007), pp. 181–196.
[7] N. Chopra and M. W. Spong, *On exponential synchronization of Kuramoto oscillators,* IEEE Trans. Automatic Control, 54 (2009), pp. 353–357.
[8] J.-G. Dong and X. Xue, *Synchronization analysis of Kuramoto oscillators,* Commun. Math. Sci., 11 (2013), pp. 465–480.
[9] F. Dörfler and F. Bullo, *Synchronization in complex networks of phase oscillators: A survey,* Automatica, 50 (2014), pp. 1539–1564.
[10] F. Dörfler and F. Bullo, *On the critical coupling for Kuramoto oscillators,* SIAM. J. Appl. Dyn. Syst., 10 (2011), pp. 1070–1099.
[11] A. Gushchin, E. Mallada and A. Tang, *Synchronization of phase-coupled oscillators with plastic coupling strength,* Proceedings of Information Theory and Applications Workshop. 2015.
[12] S.-Y. Ha, T. Y. Ha and J.-H. Kim, *On the complete synchronization for the globally coupled Kuramoto model,* Physica D, 239 (2010), pp. 1692–1700.
[13] S.-Y. Ha, H. K. Kim and J. Park, *Remarks on the complete synchronization of Kuramoto oscillators,* Nonlinearity, 28 (2015), pp. 1441–1462.
[14] S.-Y. Ha, H. K. Kim and S. W. Ryoo, *Emergence of phase-locked states for the Kuramoto model in a large coupling strength,* To appear in Commun. Math. Sci.
[15] S.-Y. Ha, Z. Li and X. Xue, *Formation of phase-locked states in a population of locally interacting Kuramoto oscillators,* J. Differential Equations, 255 (2013), pp. 3053–3070.
[16] D. O. Hebb, *The organization of behavior,* Wiley, New York 1949.
[17] A. Jadbabaie, N. Motee and M. Barahona, *On the stability of the Kuramoto model of coupled nonlinear oscillators,* Proceedings of the American Control Conference 4296–4301, 2004.
[18] Y. Kuramoto, *Chemical Oscillations, waves and turbulence,* Springer-Verlag, Berlin. 1984.
[19] Y. Kuramoto, *International symposium on mathematical problems in mathematical physics,* Lecture Notes in Theoretical Physics, 30 (1975), pp. 420.
[20] R. E. Mirollo and S. H. Strogatz, *Synchronization of Pulse-Coupled Biological Oscillators,* SIAM J. Appl. Math., 50 (1990), pp. 1645–1662.
[21] R. K. Niyogi and L. Q. English, *Learning-rate-dependent clustering and self-development in a network of coupled phase oscillators,* Phys. Rev. E, 80 (2009), pp. 066213.
[22] A. Pikovsky, M. Rosenblum and J. Kurths, *Synchronization: A universal concept in nonlinear sciences,* Cambridge University Press, Cambridge, 2001.
[23] Q. Ren and J. Zhao, *Adaptive coupling and enhanced synchronization in coupled phase oscillators,* Phys. Review E, 76 (2007), pp. 016207.
[24] L. Scardovi, *Clustering and synchronization in phase models with state dependent coupling*, 49th IEEE Conference on Decision and Control, December 15-17, 2010, Hilton Atlanta Hotel, Atlanta, GA, USA.
[25] P. Seliger, S. C. Young and L. S. Tsimring, *Plasticity and learning in a network of coupled phase oscillators,* Phys. Review E, 65 (2002), pp. 041906.
[26] S. H. Strogatz, *From Kuramoto to Crawford: exploring the onset of synchronization in populations of coupled oscillators,* Physica D, 143 (2000), pp. 1–20.
[27] D. Taylor, E. Ott and J. G. Restrepo, *Spontaneous synchronization of coupled oscillator systems with frequency adaptation,* Phys. Rev. E, 81 (2010), pp. 046214.
[28] J. L. van Hemmen and W. F. Wreszinski, *Lyapunov function for the Kuramoto model of nonlinearly coupled oscillators,* J. Stat. Phys., 72 (1993), pp. 145–166.
[29] A. T. Winfree, *Biological rhythms and the behavior of populations of coupled oscillators,* J. Theor. Biol., 16 (1967), pp. 15–42.

(Seung-Yeal Ha)
Department of Mathematical Sciences and Research Institute of Mathematics
Seoul National University, Seoul 151-747, Korea
E-mail address: syha@snu.ac.kr
URL: http://www.math.snu.ac.kr/~syha

(Se Eun Noh)
Department of Mathematics
Myongji University, Yong-In, 449-728, Korea
E-mail address: senoh@mju.ac.kr

(Jinyeong Park)
Department of Mathematical Sciences,
Seoul National University, Seoul 151-747, Korea
E-mail address: pjy40@snu.ac.kr